WORKING NOTES ON FLOW EQUATIONS FOR CMPS

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Abstract. In these notes we record all the intermediate derivations for the flow equations approach for cMPS.

1. Introduction

Recall that our variational class is defined, in one dimension, for a system of length $L$ in terms of the evolution of a continuously measured auxiliary system $\mathcal{B}$. This, in turn, is given by the propagator

\begin{equation}
U(L) = \mathcal{T} e^{i \int_0^L G(s) \, ds},
\end{equation}

where $G(s)$ is given by the limit $\lim_{n \to \infty} G_n(s)$, with

\begin{equation}
G_n(s) = K(s) + \sqrt{\epsilon} \sum_{j=1}^n \delta(s - j\epsilon)(i R(s) \otimes a_j^\dagger - i R^\dagger(s) \otimes a_j),
\end{equation}

and $\epsilon = L/n$. The system of $n$ meters is referred to as $12 \cdots n$. In the continuum limit $n \to \infty$ we refer to this collection as $\mathcal{A}$. The propagator is clearly defined in terms of two one-parameter operators: (i) a hermitian path $K(s)$, $s \in [0, L]$; and (ii) a path of complex matrices $R(s)$, $s \in [0, L]$.

Our variational class (of the system+auxiliary system collection) is then defined by

\begin{equation}
|\Phi(L)\rangle = U|\Omega\rangle,
\end{equation}

where $|\Omega\rangle \equiv |\Omega\rangle_{A\mathcal{B}}$ is the vacuum state of the system+auxiliary system. We now make an important simplification: we assume translation invariance, which we impose by assuming $K(s) = K$ and $R(s) = R$, $s \in [0, L]$.

We want to solve the (real- or imaginary-time) dynamics for a system with hamiltonian $H$ (we focus on imaginary-time evolution):

\begin{equation}
\frac{d}{d\beta} |\Phi(\beta)\rangle = -H \otimes I_{\mathcal{B}} |\Phi(\beta)\rangle.
\end{equation}

We are going to simulate this dynamics by allowing finding a path operators $K(\beta)$ and $R(\beta)$ which best approximates the dynamics.

As we know, the reduced density operator $\rho_\mathcal{B}(t)$ for the auxiliary system defines the quantum state of the quantum field. (From now on we drop the subscript $\mathcal{B}$ on $\rho_\mathcal{B}(s)$; we understand henceforth, unless otherwise stated, that the symbol $\rho$ is the state of the auxiliary system.)
system.) In order that physical observables arise from a legal quantum state we require that 
\( \rho_3(t) \) satisfies the Lindblad equation

\[
\frac{d\rho(s)}{ds} = -i[K, \rho(s)] - \frac{1}{2} \left( R^\dagger R \rho(s) - 2R \rho(s) R^\dagger + \rho(s) R^\dagger R \right).
\]

At this point it becomes slightly more convenient to move to a different representation for the 
state of our auxiliary system. This representation, known as the \textit{Jamiolkowski isomorphism} 
within the quantum information literature, is simply just another notation to represent the 
hilbert space of operators (with the \textit{hilbert-schmidt} inner product \( (A, B) \equiv \text{tr}(A^\dagger B) \)) on a 
hilbert space, and is defined as follows:

\[
A = \sum_{j,k=1}^{D} a_{jk} |j\rangle \langle k| \implies |A \rangle = \sum_{j,k=1}^{D} a_{jk} |j\rangle |k\rangle.
\]

It is easy to see that the operation of “multiplication on the left by \( A \)” , \( L_A(B) \equiv AB \), becomes, in this notation

\[
L_A(B) \mapsto A \otimes I |B \rangle.
\]

The operation of “multiplication on the right by \( A \)” , \( R_A(B) \equiv BA \), becomes, in this

\[
R_A(B) \mapsto I \otimes A^T |B \rangle.
\]

In terms of this new notation the Lindblad equation (1.5) becomes

\[
\frac{d}{ds} |\rho(s) \rangle = \left[ -i \left( K \otimes I - I \otimes K^T \right) - \frac{1}{2} \left( R^\dagger R \otimes I - 2R \otimes R^\dagger + I \otimes R^T R \right) \right] |\rho(s) \rangle.
\]

Now we allow \( K \) and \( R \) to depend on an auxiliary parameter \( \beta \); we write

\[
M(\beta) \equiv -i \left( K(\beta) \otimes I - I \otimes K^T(\beta) \right) - \frac{1}{2} \left( R^\dagger(\beta) R(\beta) \otimes I - 2R(\beta) \otimes R^\dagger(\beta) + I \otimes R^T(\beta) R(\beta) \right)
\]

so that

\[
|\rho(t) \rangle = e^{tM(\beta)} |\rho(0) \rangle.
\]

Let’s make some comments on the generator \( M(\beta) \). When this operator is diagonalizable it is 
known to have a spectral decomposition

\[
M(\beta) = S^{-1}(\beta) \left( \sum_{j=1}^{D} m_j(\beta) |j\rangle \langle j| \right) S(\beta),
\]

where the eigenvalues \( m_j(\beta) \) are assumed to be in arranged in decreasing real part, such that 
\( m_1(\beta) = 0 \). Generically it is the case that \( \Re(m_2(\beta)) < 0 \) and

\[
\lim_{t \to \infty} e^{tM(\beta)} = S^{-1}(\beta) |1\rangle \langle 1| S(\beta) = |I\rangle \langle \rho(\beta)|,
\]

where

\[
|I \rangle = \sum_{j=1}^{D} |jj\rangle.
\]
To proceed with this calculation we notice that, in the limit $L \to \infty$, we can express the expectation value $\langle \Phi(\beta) | H | \Phi(\beta) \rangle$, for the Lieb-Liniger model in a relatively simple form:

\begin{equation}
E(\beta) = \langle \Phi(\beta) | H | \Phi(\beta) \rangle = \text{tr}(\{K(\beta), R(\beta)\} [K(\beta), R(\beta)] \rho(\beta)) + c \text{tr}(R^2(\beta) \rho(\beta) R^\dagger(\beta)^2),
\end{equation}

where $\rho(\beta)$ is the (unique) steady state of the Lindblad equation defining $|\Phi(\beta)\rangle$.

Now, to compute $E'(\beta)$ we need to compute $\frac{d\rho(\beta)}{d\beta}$ it is convenient to use the Jamiolkowski notation:

\begin{equation}
\frac{d\rho(\beta)}{d\beta} \leftrightarrow \frac{d}{d\beta}|\rho(\beta)\rangle.
\end{equation}

We next exploit the assumed fact that $|\rho(\beta)\rangle$ is the unique eigenvector of eigenvalue 0 of the operator

\begin{equation}
M(\beta) \equiv -i \left(K(\beta) \otimes \mathbb{1} - \mathbb{1} \otimes K^T(\beta)\right) - \frac{1}{2} \left(R^\dagger(\beta) R(\beta) \otimes \mathbb{1} - 2 R(\beta) \otimes R^\dagger(\beta) + \mathbb{1} \otimes R^T(\beta) R(\beta)\right).
\end{equation}

Realising this allows us to immediately compute the derivative using standard perturbation theory:

\begin{equation}
\frac{d}{d\beta}|\rho(\beta)\rangle = -\frac{\mathbb{1}}{M(\beta)} \frac{dM(\beta)}{d\beta} |\rho(\beta)\rangle + |\rho(\beta)\rangle \mathbb{1} \frac{dM(\beta)}{d\beta} |\rho(\beta)\rangle.
\end{equation}

3. Computation of ground-state properties via flow equations

In this section we show how to use the technology developed in the previous sections to calculate the ground-state properties for the Lieb-Liniger model via the flow-equations approach. Recall that this procedure works by solving for the derivatives of the parameters $K'(\beta)$ and $R'(\beta)$ defining the state of the quantum field by working out the direction which minimises the corresponding change in $E(\beta)$ the fastest. This is achieved by solving the following optimisation problem

\begin{equation}
\min_{K'(\beta), R'(\beta)} \frac{dE(\beta)}{d\beta} + \lambda \text{tr}(K'(\beta)^2) + \mu \text{tr}(R'(\beta) R'(\beta))
\end{equation}

such that $\text{tr}(R^\dagger(\beta) R(\beta) \rho(\beta)) = \varrho$,

where $\lambda$ and $\mu$ are lagrange multipliers chosen to enforce the boundedness of the derivatives $K'(\beta)$ and $R'(\beta)$ and $\varrho$ is the desired particle density. In order to ensure that the particle density remains constant (at say, $\varrho = 1$) we can impose the density constraint by adding a further lagrange multiplier, so that the $K'$ and $R'$ are given by the solution to

\begin{equation}
\min_{K'(\beta), R'(\beta)} \frac{dE(\beta)}{d\beta} + \lambda \text{tr}(K'(\beta)^2) + \mu \text{tr}(R'(\beta) R'(\beta)) + \nu \frac{d}{d\beta} \text{tr}(R^\dagger(\beta) R(\beta) \rho(\beta)).
\end{equation}

(We now drop the explicit dependence on $\beta$.) We thus study the lagrangian

\begin{equation}
L(K', R'; \lambda, \mu, \nu) = \frac{d}{d\beta} \text{tr} \left( \left[ (Q, R) [Q, R] + c R^2 R^\dagger + \nu R^\dagger R \right] \rho \right) + \lambda \text{tr}(K'^2) + \mu \text{tr}(R'^\dagger R'),
\end{equation}
where $Q = -\frac{1}{2} R^\dagger R - i K$. Thus we are reduced to solving the \textit{unconstrained} hamiltonian whose image is $H = [Q, R]^\dagger [Q, R] + c R^\dagger R^2 + \nu R^\dagger R$ so that $H = H_{LL} + \nu H_D$, with $H_D = R^\dagger R$.

Writing $K = \sum_{j,k=1}^D K_{jk} |j\rangle \langle k|$ and $R = \sum_{j,k=1}^D R_{jk} |j\rangle \langle k|$, the equations for the extrema are given by

\begin{align*}
\frac{d}{dK_{jk}'} L(K', R'; \lambda, \mu, \nu) &= 0, \quad j, k = 1, 2, \ldots, D, \quad (3.4) \\
\frac{d}{dR_{jk}'} L(K', R'; \lambda, \mu, \nu) &= 0, \quad j, k = 1, 2, \ldots, D, \quad \text{and} \quad (3.5) \\
\frac{d}{d\nu} L(K', R'; \lambda, \mu, \nu) &= 0. \quad (3.6)
\end{align*}

(The extremal equations for $\lambda$ and $\mu$ simply ensure that $K'$ and $R'$ are bounded.)

The next step is to explicitly calculate and solve (3.4), (3.5), and (3.6). Both (3.4) and (3.5) require that we calculate

\begin{align*}
(\text{I}) &= \frac{d}{d\beta} \left( [Q, R]^\dagger [Q, R] + c R^\dagger R^2 + \nu R^\dagger R \right) \\
(\text{II}) &= \frac{d}{d\beta} \rho.
\end{align*}

For the term (I) we find

\begin{align*}
(\text{I}) &= [R^\dagger, Q'] C + [R^\dagger, Q'] C + C'^\dagger [Q', R] + C'^\dagger [Q, R'] + c R^\dagger R^2 + c R^\dagger R^\dagger R^2 + c R^\dagger R' R + c R^\dagger^2 R R' + \nu R^\dagger R + \nu R^\dagger R',
\end{align*}

where $C = [Q, R]$ and

\begin{align*}
Q' &= -\frac{1}{2} R^\dagger R - \frac{1}{2} R^\dagger R' - i K'. \quad (3.9)
\end{align*}

The second term is given, in Jamiolkowski notation, by (2.5):

\begin{align*}
(\text{II}) &= -\frac{1}{M} M' |\rho\rangle + |\rho\rangle \left\langle \frac{1}{M} M' |\rho\rangle, \quad (3.10)
\end{align*}

where the equation for $M'$ is, likewise,

\begin{align*}
M' &= -i \left( K' \otimes I - I \otimes K'^T \right) - \\
&\quad \frac{1}{2} \left( R^\dagger R \otimes I + R^\dagger R' \otimes I - 2 R' \otimes R - 2 R \otimes R' + I \otimes R^T R + I \otimes R^T R' \right). \quad (3.11)
\end{align*}

We now turn to (3.4): we use the fact that

\begin{align*}
\frac{d}{dK_{jk}'} (\text{I}) &= i [R^\dagger, |j\rangle \langle k|] C - i C'^\dagger [|j\rangle \langle k|, R] \\
\text{and} \\
\frac{d}{dK_{jk}'} (\text{II}) &= -i \left(- I + |\rho\rangle \left\langle |\rho| \right. \right) \frac{1}{M} \left(|j\rangle \langle k| \otimes I - I \otimes |k\rangle \langle j| \right) |\rho\rangle, \quad (3.13)
\end{align*}
to find

\begin{equation}
\frac{d}{dR_{jk}}L(K', R'; \lambda, \mu, \nu) = \text{tr} \left[ (i[R^\dagger, |j\rangle \langle k|] C - iC^\dagger |j\rangle \langle k|, R] \rho \right] - i\left( -\langle H | + \langle H |\rho \rangle \|I\| \right) \frac{I}{M} \left( |j\rangle \langle k| \otimes I - I \otimes |k\rangle \langle j| \right) |\rho\rangle + 2\lambda \text{tr}(|j\rangle \langle k| K') = 0,
\end{equation}

where $|\Xi\rangle = -|H\rangle + \langle \rho |H\rangle |\|I\|$.

By exploiting the cyclic rule of trace we can simplify this equation to

\begin{equation}
-2\lambda K' = (i[C\rho, R^\dagger] - i[R, \rho C^\dagger]) - i \text{tr}_2 \left( |\rho\rangle \langle \Xi | \frac{I}{M} \right) + i \text{tr}_1 \left( |\rho\rangle \langle \Xi | \frac{I}{M} \right)^T.
\end{equation}

Arbitrarily setting $\lambda = 1/2$ (which, without loss of generality, ensures that the derivative is bounded) we finally obtain

\begin{equation}
K' = -i[C\rho, R^\dagger] + i[R, \rho C^\dagger] + i \text{tr}_2 \left( |\rho\rangle \langle \Xi | \frac{I}{M} \right) - i \text{tr}_1 \left( |\rho\rangle \langle \Xi | \frac{I}{M} \right)^T.
\end{equation}

It is convenient, in the sequel, to separate $K' = K'_0 + \nu K'_1$ into two components: $K'_0$, which is the part independent of $\nu$, and $K'_1$ which multiplies $\nu$, where

\begin{equation}
K'_0 = -i[C\rho, R^\dagger] + i[R, \rho C^\dagger] + i \text{tr}_2 \left( |\rho\rangle \langle \Xi_0 | \frac{I}{M} \right) - i \text{tr}_1 \left( |\rho\rangle \langle \Xi_0 | \frac{I}{M} \right)^T,
\end{equation}

and

\begin{equation}
K'_1 = i \text{tr}_2 \left( |\rho\rangle \langle \Xi_1 | \frac{I}{M} \right) - i \text{tr}_1 \left( |\rho\rangle \langle \Xi_1 | \frac{I}{M} \right)^T,
\end{equation}

with

\begin{equation}
|\Xi_0\rangle = -|H_{LL}\rangle + \langle \rho |H_{LL}\rangle |\|I\|, \quad \text{and} \quad |\Xi_1\rangle = -|H_D\rangle + \langle \rho |H_D\rangle |\|I\|.
\end{equation}

We now turn to the calculation of (3.5). Similar to the calculation for $K'$ we begin with

\begin{equation}
\frac{d}{dR_{jk}}(I) = [|k\rangle \langle j|, Q^\dagger] C - \frac{1}{2}[R^\dagger, |k\rangle \langle j|, R] C - \frac{1}{2} C^\dagger [R^\dagger, |k\rangle \langle j|, R] + c|k\rangle \langle j| R^\dagger R^2 + cR^\dagger |k\rangle \langle j| R^2 + \nu |k\rangle \langle j| R
\end{equation}

and

\begin{equation}
\frac{d}{dR_{jk}}(\Pi) = -\frac{1}{2} (-I + |\rho\rangle \langle I|) \frac{I}{M} \left( |k\rangle \langle j| R \otimes I - 2R \otimes |j\rangle \langle k| + I \otimes R^\dagger |j\rangle \langle k| \right) |\rho\rangle,
\end{equation}
This can be simplified, using the cyclic rule of trace, to
\begin{equation}
\frac{d}{dR_{jk}} L(K', R'; \lambda, \mu, \nu) = \text{tr} \left[ \left( \langle k | R | k \rangle - \frac{1}{2} R \langle k | R | k \rangle C - \frac{1}{2} C^\dagger \langle k | R | k \rangle R + c \langle k | R^1 R^2 + c R^1 \langle k | R^2 + \nu | k \rangle \langle R | R \rangle \rho \right) - \frac{1}{2} \left( -\langle H | + \langle H | \rho \rho \rangle \right) \frac{I}{M} \left( \langle k | R \otimes I - 2 R \otimes | k \rangle \langle k | + I \otimes R^T | k \rangle \right) | \rho \rangle + \mu \text{tr}(|k \rangle \langle k | R' \rangle) = 0.
\end{equation}

This can be further reduced to
\begin{equation}
- \mu \text{tr}(|k \rangle \langle k | R') = \text{tr} \left[ |k \rangle \langle j | \left( [Q^\dagger, C \rho] - \frac{1}{2} R [C \rho, R^1] - \frac{1}{2} R [R, \rho C^\dagger] + c R^1 R^2 \rho + c R^1 \rho R^1 + \nu R \rho \right) \right] - \frac{1}{2} \text{tr} \left( [k | R \otimes I - 2 R \otimes | j \rangle \langle j | + I \otimes R^T | j \rangle \right) | \rho \rangle \langle \Xi | \frac{I}{M} \right].
\end{equation}

which becomes, after setting \( \mu = 1 \),
\begin{equation}
R' = -[Q^\dagger, C \rho] + \frac{1}{2} R [C \rho, R^1] + \frac{1}{2} R [R, \rho C^\dagger] - c R^1 R^2 \rho - c R^2 \rho R^1 - \nu R \rho + \frac{1}{2} R \text{tr} \left( | \rho \rangle \langle \Xi | \frac{I}{M} \right) - \text{tr} \left( R \otimes I | \rho \rangle \langle \Xi | \frac{I}{M} \right) \right)^T + \frac{1}{2} R \text{tr} \left( | \rho \rangle \langle \Xi | \frac{I}{M} \right)^T.
\end{equation}

As before, as in the case of \( K' \), it is convenient to separate \( R' = R'_0 + \nu R'_1 \) into two pieces, where
\begin{equation}
R'_0 = -[Q^\dagger, C \rho] + \frac{1}{2} R [C \rho, R^1] + \frac{1}{2} R [R, \rho C^\dagger] - c R^1 R^2 \rho - c R^2 \rho R^1 + \frac{1}{2} R \text{tr} \left( | \rho \rangle \langle \Xi_0 | \frac{I}{M} \right) - \text{tr} \left( R \otimes I | \rho \rangle \langle \Xi_0 | \frac{I}{M} \right) \right)^T + \frac{1}{2} R \text{tr} \left( | \rho \rangle \langle \Xi_0 | \frac{I}{M} \right)^T,
\end{equation}

and
\begin{equation}
R'_1 = -R \rho + \frac{1}{2} R \text{tr} \left( | \rho \rangle \langle \Xi_1 | \frac{I}{M} \right) - \text{tr} \left( R \otimes I | \rho \rangle \langle \Xi_1 | \frac{I}{M} \right) \right)^T + \frac{1}{2} R \text{tr} \left( | \rho \rangle \langle \Xi_1 | \frac{I}{M} \right)^T.
\end{equation}
The final equation (3.6) ensures that the derivatives preserve the particle density: this can be solved for $\nu$ as follows

\[
\frac{d}{d\nu} L(K', R'; \lambda, \mu, \nu) = \text{tr} \left[ (R_0' + \nu R_1')^\dagger R \rho \right] + \text{tr} \left[ R_1^\dagger (R_0' + \nu R_1') \rho \right] + \text{tr} \left[ R_1^\dagger R (\rho_0 + \nu \rho_1') \right] = 0,
\]
where, in Jamiołkowski notation,

\[
|\rho_0\rangle = -\frac{1}{M} M_0^\dagger |\rho\rangle + |\rho\rangle \langle \frac{1}{M} M_0^\dagger |\rho\rangle
\]
and

\[
|\rho_1\rangle = -\frac{1}{M} M_1^\dagger |\rho\rangle + |\rho\rangle \langle \frac{1}{M} M_1^\dagger |\rho\rangle,
\]
with

\[
M_0' = -i \left( K_0' \otimes \mathbb{I} - \mathbb{I} \otimes K_0'^T \right) - \frac{1}{2} \left( R_0' R \otimes \mathbb{I} + R_1^\dagger R_0 \otimes \mathbb{I} - 2R_0' \otimes \mathbb{I} - 2R_1^\dagger \otimes \mathbb{I} + \mathbb{I} \otimes R_0'^T \mathbb{I} + \mathbb{I} \otimes R_1'^T \mathbb{I} \right).
\]

and

\[
M_1' = -i \left( K_1' \otimes \mathbb{I} - \mathbb{I} \otimes K_1'^T \right) - \frac{1}{2} \left( R_1^\dagger R \otimes \mathbb{I} + R_1' R_0 \otimes \mathbb{I} - 2R_1' \otimes \mathbb{I} - 2R_0' \otimes \mathbb{I} + \mathbb{I} \otimes R_1'^T \mathbb{I} + \mathbb{I} \otimes R_0'^T \mathbb{I} \right).
\]

This allows us to obtain the following solution for $\nu$:

\[
\nu = -\frac{\text{tr} \left[ R_0^\dagger R \rho \right] + \text{tr} \left[ R_1^\dagger R_0' \rho \right] + \text{tr} \left[ R_1^\dagger R_0' \rho \right]}{\text{tr} \left[ R_0^\dagger R \rho \right] + \text{tr} \left[ R_1^\dagger R_0' \rho \right] + \text{tr} \left[ R_1^\dagger R_0' \rho \right]}.
\]

With this expression for $\nu$ we now have the complete set of first-order nonlinear differential equations for $K$ and $R$:

\[
K' = f(K, R)
\]
\[
R' = g(K, R),
\]
where

\[
f(K, R) = -i[C\rho, R^\dagger] + i[R, \rho C^\dagger] + i \text{tr}_2 \left( |\rho\rangle \langle \Xi | \frac{\mathbb{I}}{M} \right) - i \text{tr}_1 \left( |\rho\rangle \langle \Xi | \frac{\mathbb{I}}{M} \right)^T
\]
and

\[
g(K, R) = -[Q^\dagger, C\rho] + \frac{1}{2} R[C\rho, R^\dagger] + \frac{1}{2} R[R, \rho C^\dagger] - cR^\dagger R^2 \rho - cR^2 \rho R^\dagger - \nu R \rho + \frac{1}{2} R \text{tr}_2 \left( |\rho\rangle \langle \Xi | \frac{\mathbb{I}}{M} \right) - \text{tr}_1 \left( R \otimes \mathbb{I} |\rho\rangle \langle \Xi | \frac{\mathbb{I}}{M} \right) + \frac{1}{2} R \text{tr}_1 \left( |\rho\rangle \langle \Xi | \frac{\mathbb{I}}{M} \right)^T,
\]
and $\nu \equiv \nu(K, R)$ is given by (3.35). These differential equations can be numerically integrated using standard methods, eg., the Runge-Kutta 4th order method.

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