Introduction to quantum information techniques for complex quantum systems

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Outline
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• Quantum mechanics
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• Complex quantum systems: spin systems
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• Hamiltonian complexity
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• Quantum mechanics
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• Simulation
• Hamiltonian complexity
• Structure of dynamics
Quantum mechanics: postulates

**Postulate 1**: associated to any isolated physical system is a *Hilbert space* known as the state space of the system. The system is completely described by its state vector, which is a *unit vector* in the system's state space.
Example

• the simplest quantum mechanical system: the qubit.

• orthonormal basis: $|0\rangle$ and $|1\rangle$

• arbitrary state: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

• We take the qubit to be our fundamental quantum system.
Quantum mechanics: postulates

Postulate 1’ (combining quantum systems). Suppose we have two separate isolated quantum systems $A$ and $B$ with state spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then the state space of the compound system $AB$ is $\mathcal{H}_1 \otimes \mathcal{H}_2$. 
Exercise

What is the state space for 3 qubits?
Quantum mechanics: postulates

Postulate 2: the evolution of a closed quantum system is described by a unitary transformation. That is, the state of the system at time $t_1$ is related to the state of the system at time $t_2$ by a unitary operator $U$ which depends only on the times $t_1$ and $t_2$:

$$|\psi'\rangle = U|\psi\rangle$$
Example

• the Pauli matrices are given by:

\[
\begin{align*}
\sigma^X &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \sigma^Y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, & \sigma^Z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{align*}
\]
Quantum mechanics: postulates

**Postulate 2’**: The time evolution of the state of a closed quantum system is described by the Schrödinger equation:

\[ i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle \]

We’ll rescale things so \( \hbar \equiv 1 \).
Exercise

Show postulate 2 and 2’ are equivalent
Postulate 3: quantum measurements are described by a collection of measurement operators $M_m$. The index $m$ refers to the measurement outcomes that may occur in the experiment. The probability that result “$m$” occurs is given by:

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle,$$

and the state of the system after the measurement is:

$$\frac{M_m | \psi \rangle}{\sqrt{p(m)}}$$
Complex quantum systems
Complex quantum systems

(Roughly speaking) a **complex quantum system** is a quantum system comprising many interacting subsystems.
Complex quantum systems

In this lecture: they are 1D quantum spin systems, which are collections of $n$ quantum spins:
Complex quantum systems

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local hilbert space: $\mathbb{C}^d$
Complex quantum systems

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In this lecture: they are 1D quantum spin systems, which are collections of $n$ quantum spins:

$n$ quantum spins with global hilbert space:

$$(\mathbb{C}^d)^\otimes n$$
Remarks

- We let spin dimension be $d=2$ for simplicity.
- Dimension of Hilbert space for $n$ spins is: $2^n$.
- Computational cost of simulating a quantum spin system is naively exponential in $n$. (Why?)
Interactions

The way our spins interact is via their nearest-neighbours:

\[
h = \begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1d^2} \\
\vdots & \ddots & \vdots \\
\alpha_{d^21} & \cdots & \alpha_{d^2d^2}
\end{pmatrix}
\]
Hamiltonian

\[ H = \sum_{j=1}^{n-1} h_j \]

where

\[ h_j = \mathbb{1}_{1\ldots j-1} \otimes h \otimes \mathbb{1}_{j+2\ldots n} \]
Example

- The *transverse Ising model*:

\[
H = \sum_{j=1}^{n-1} \sigma_j^x \sigma_{j+1}^x + \lambda \sum_{j=1}^{n} \sigma_j^z
\]
Important numbers: $n$, $d$, and $||h||$

- size
- energy normalisation
Why 1D spin systems?

Although seemingly restricted, the dynamics of an appropriately engineered 1D quantum spin system can simulate (with some overhead) the dynamics of any locally interacting complex quantum system. (This is a hard, but now classical, result from quantum information.)
The state of the system
The state of the system:

- When the system is in equilibrium at temperature $T$ it is in the *Gibbs state*
  \[ \rho = \frac{e^{-\beta H}}{Z} \]
  where $\beta = 1/(k_B T)$ and $Z = \text{tr}(e^{-\beta H})$

- At zero temperature it is in the *ground state* (assumed non-degenerate):
  \[ |\Omega\rangle \langle \Omega| = \lim_{\beta \to \infty} \frac{e^{-\beta H}}{Z} \]
The state of the system:

- When the system is out of equilibrium it evolves according to:

\[ |\psi(t)\rangle = e^{itH} |\psi(0)\rangle \]
Observables

• Physical observations provide information on the expectation value:
  \[ \langle A \rangle = \text{tr}(\rho A) \]
  where \( A \) is a hermitian operator

• For quantum spin systems observables have bounded support \( S \):
  \[ A \equiv A_S \otimes \mathbb{I}_{[n] \setminus S} \]
  i.e., \( |S| = O(1) \). \( |S| \) is another important number.
Physically measurable observables have small support: $|S| = 1$ or $2$

Eg. 2-point correlation function:

$$A = A_1 \otimes I_2 \cdots j-1 \otimes A_j \otimes I_{j+1} \cdots n$$
Simulation
Simulation
What is simulation?

- Simulation means: *make a theoretical prediction*
- What counts as a *simulation*?
  - Theoretical calculations
  - Computer-assisted proof
  - Numerical solution of equations
The simulation Problem

In physics we want to simulate the outcomes of physical observations:

**The simulation problem**

**Input:**
1. an observable $A$
2. a tolerance $\epsilon$

**Output:** an approximation $\alpha$ s.t.

$$|\alpha - \langle A \rangle| \leq \epsilon$$
Hamiltonian Complexity
How *hard* is it to simulate a physical system?
The simulation problem

- In theoretical physics one is usually happy if the answer is *physically correct*. (i.e., it is not so important that the answer is strictly close to the correct answer to some prespecified tolerance $\epsilon$)

- In hamiltonian complexity it *matters* if the prediction is within the tolerance. Thus: “*theoretical physics with error bars*”. 
The simulation problem

• Several variants of the simulation problem:
  • Equilibrium:
    • Non-zero temperature
    • Zero temperature
  • Non-equilibrium
    • Dynamical observables
The equilibrium simulation problem

*Input:* 
(i) a Hamiltonian $H$, and size $n$
(ii) an inverse temperature $\beta$
(iii) an observable $A$
(iv) a tolerance $\epsilon$

*Output:* an approximation $\alpha$ s.t.

$$\left| \text{tr} \left( A \frac{e^{-\beta H}}{Z} \right) - \alpha \right| \leq \epsilon$$
Non-Equilibrium simulation Problem

The non-equilibrium simulation problem

**Input:**
(i) a hamiltonian $H$, and size $n$
(ii) an initial state $|\psi(0)\rangle$
(iii) a time $t$
(iv) an observable $A$
(v) a tolerance $\epsilon$

**Output:** an approximation $\alpha$ s.t.

$$|\langle\psi(t)|A|\psi(t)\rangle - \alpha| \leq \epsilon$$
Simulating non-equilibrium phenomena
Theorem. 
If 
\[ |t| \leq O(\log(n)) \]
and
\[ |\psi(t)\rangle = e^{itH} |\psi(0)\rangle \]
then
\[ \langle \psi(t)|A|\psi(t)\rangle \]
can be well-approximated with
\[ \text{poly}(n, 1/\epsilon, 2^{\text{supp}(A)}) \]
resources. (A is some observable.)

Argument

• Uses a reformulation of a result known as the **Lieb-Robinson bound**.
The Lieb-Robinson bound

Proposition LR. Let $A$ be a hermitian operator with $\text{supp}(A) = O(I)$. Then

$$\|A(t) - A_\Lambda(t)\|_\infty \leq ce^{-v|\Lambda| + k|t|}$$

where $v$ and $k$ are constants, and

$$A_\Lambda(t) = e^{-itH_\Lambda} Ae^{itH_\Lambda}$$

and

$$H_\Lambda = \sum_{\text{supp}(h_j) \subset \Lambda} h_j$$

with $\Lambda \subset [n]$ centred on $\text{supp}(A)$.
The Lieb-Robinson bound in Pictures

\[ A(t) = e^{itH} \]
The Lieb-Robinson bound in Pictures
The Lieb-Robinson bound in Pictures

\[ A(t) \approx A^\Lambda(t) \]
Proof of result

• Idea is to divide the chain into two pieces $A$ and $B$: and approximate $e^{itH}$ with $e^{itH_A} \otimes e^{itH_B}$

• This approximation is bad near the cut point, so we fix it up with the unitary $V(t)$ defined by:

$$e^{itH} = (e^{itH_A} \otimes e^{itH_B})V(t)$$
The cut approximation

• The unitary $V(t)$ satisfies T-D Schrödinger equation

$$\frac{dV(t)}{dt} = iV(t)h_I(t)$$

where

$$h_I(t) = e^{-itH} h_I e^{itH}$$

• We use the Lieb-Robinson bound to approximate:

$$h_I(t) \approx h_I^\wedge(t) = e^{-itH^\wedge} h_I e^{itH^\wedge}$$

and then define

$$\frac{dV_\wedge(t)}{dt} = iV_\wedge(t)h_I^\wedge(t)$$
Note that:

\[ \| V(t) - V_\Lambda(t) \| \leq |t|e^{-v|\Lambda|+k|T|} \]

AND

\[ \text{supp}(V_\Lambda(t)) = \Lambda \]
In pictures: a recursion

\[ e^{itH} \approx e^{itH_A} V_{A}(t) e^{itH_B} \]

\(|A|\)
In pictures: a recursion

\[ e^{itH} \approx e^{itH_A} e^{itH_B} \]

\[ V_\Lambda(t) \]

\[ V_\Lambda(t) \]

\[ V_\Lambda(t) \]
Recursion: final result

Chasing epsilons through shows that, to ensure tolerance of $\epsilon$ we need $|\Lambda| = c \log(n/\epsilon) + |t|$
Lecture 1: summary

- Introduced class of complex quantum systems: strongly interacting quantum spin systems.
- Introduced the simulation problem.
- Showed that dynamics of any strongly interacting quantum spin system can be efficiently simulated.