Q1 Manifolds:

(i) Is a torus a manifold? Why or why not?

(ii) Is a figure 8 a manifold? Why or why not?

(iii) Take the two dimensional manifold embedded in \( \mathbb{R}^3 \) defined by the equation
\[
z - x^2 - y^2 = 0.
\] Is there a single chart for this manifold or does one need multiple charts?

(iv) The real projective space \( \mathbb{RP}^n \) is the set of lines through the origin in \( \mathbb{R}^{n+1} \). Two points \( \vec{x}, \vec{y} \in \mathbb{R}^{n+1} \) define the same line if \( \vec{x} = \alpha \vec{y} \) with \( \alpha \neq 0 \) and \( \vec{x}, \vec{y} \neq 0 \). For each line, we can choose a point \( \vec{x} \in \mathbb{R}^{n+1} \) as a representative.

For each \( i \in \{1, \ldots, n+1\} \), we define \( U_i \) to be the set of lines with \( x_i \neq 0 \). Then we define coordinate maps on each \( U_i \) by
\[
\varphi_i : U_i \rightarrow \mathbb{R}^n
\]
\[
\varphi_i : (x_1, \ldots, x_{n+1}) \rightarrow \left( \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i} \right),
\]
which is well defined because \( x_i \neq 0 \) when \( \vec{x} \in U_i \). Notice that \( \varphi_i \) always takes two points on the same line in \( \mathbb{RP}^n \) to the same point in \( \mathbb{R}^n \), so the map is independent of which \( \vec{x} \) we choose as our representative of each line.

For \( \vec{x} \in U_i \cap U_j \), find the coordinate transformation \( \varphi_i \circ \varphi_j^{-1} \).

What is \( \varphi_1 \circ \varphi_2^{-1} \) for \( \mathbb{RP}^1 \)? Is this continuous?

Q2 Atlases:

(i) Suppose our manifold is a circle \( S^1 \) embedded in \( \mathbb{R}^2 \) via the equation \( x^2 + y^2 = 1 \).

Let us define an atlas with two charts with coordinate maps \( \varphi_1 \) and \( \varphi_2 \) via
\[
\varphi_1^{-1} : (0, 2\pi) \rightarrow S^1
\]
\[
\varphi_1^{-1} : \theta \rightarrow (\cos(\theta), \sin(\theta))
\]
and
\[
\varphi_2^{-1} : (-\pi, \pi) \rightarrow S^1
\]
\[
\varphi_2^{-1} : \theta \rightarrow (\cos(\theta), \sin(\theta)).
\]

Notice that \( \varphi_i^{-1} \) have different domains and are invertible.

Verify that the maps \( \varphi_1 \circ \varphi_2^{-1} \) and \( \varphi_2 \circ \varphi_1^{-1} \) are smooth.
(ii) Stereographic coordinates for the sphere work as follows. Imagine the sphere embedded in $\mathbb{R}^3$ via the constraint $x^2 + y^2 + z^2 = 1$. Then we join the North Pole $(0, 0, 1)$ to any other point $(x, y, z)$ on the sphere by a straight line and continue that line until it intersects with the $z = 0$ plane $(X, Y, 0)$. Show that the point of intersection is given by

$$X = \frac{x}{1-z}, \quad Y = \frac{y}{1-z}. \quad (5)$$

Mapping points on the sphere to the $z = 0$ plane in this way gives us a coordinate map. Can we use this coordinate map to define a chart covering the entire sphere?

(iii) Suppose we use polar coordinates for the sphere $\theta$ and $\phi$, such that the points on the sphere are given by

$$x = \sin \theta \cos \phi, \quad y = \sin \theta \sin \phi, \quad z = \cos \theta. \quad (6)$$

Show that polar and stereographic coordinates are related via

$$X = \cot \frac{\theta}{2} \cos \phi, \quad Y = \cot \frac{\theta}{2} \sin \phi. \quad (7)$$

Do polar coordinates suffice for a single chart covering the entire sphere?

Q3 Tangent vectors:

(i) Show that all tangent vectors at a point $p$ in an $n$ dimensional manifold are a vector space.

(ii) Suppose we have the path $\gamma(t) : (a, b) \to \mathcal{M}$, with $(a, b) \subset \mathbb{R}$ and $\gamma(0) = p \in \mathcal{M}$. We also have coordinate functions $q^i$, such that $q^i(\gamma(t)) = x^i(t)$. Then we have the tangent vector $X = X^i \frac{\partial}{\partial x^i} \in T\mathcal{M}_p$, with

$$X^i = \frac{dx^i}{dt}. \quad (8)$$

(Really we should write $x^i(t)$ instead of $x^i$.)

If $t$ is interpreted as time and $\gamma(t)$ is a particle’s position, then how would you interpret $X[x^i(t)]$?