Symplectic Geometry and Classical Mechanics

Exercise Sheet 1: Manifolds and tangent spaces

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Q1 Manifolds:

(i) Is a torus a manifold? Why or why not?

(ii) Is a figure 8 a manifold? Why or why not?

(iii) Take the two dimensional manifold embedded in $\mathbb{R}^3$ defined by the equation

$$z - x^2 - y^2 = 0. \tag{1}$$

Is there a single chart for this manifold or does one need multiple charts?

(iv) The real projective space $\mathbb{R}P^n$ is the set of lines through the origin in $\mathbb{R}^{n+1}$. Two points $\vec{x}, \vec{y} \in \mathbb{R}^{n+1}$ define the same line if $\vec{x} = \alpha\, \vec{y}$ with $\alpha \neq 0$ and $\vec{x}, \vec{y} \neq 0$. For each line, we can choose a point $\vec{x} \in \mathbb{R}^{n+1}$ as a representative.

For each $i \in \{1, ..., n+1\}$, we define U_i to be the set of lines with $x_i \neq 0$. Then we define coordinate maps on each U_i by

$$\begin{aligned} \varphi_i :\ & U_i \to \mathbb{R}^n \\ \varphi_i :\ & (x_1, ..., x_{n+1}) \to \left(\frac{x_1}{x_i}, ... \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_{n+1}}{x_i} \right), \end{aligned} \tag{2}$$

which is well defined because $x_i \neq 0$ when $\vec{x} \in U_i$. Notice that φ_i always takes two points on the same line in $\mathbb{R}P^n$ to the same point in $\mathbb{R}^n$, so the map is independent of which $\vec{x}$ we choose as our representative of each line.

For $\vec{x} \in U_i \cap U_j$, find the coordinate transformation $\varphi_i \circ \varphi_j^{-1}$.

What is $\varphi_1 \circ \varphi_2^{-1}$ for $\mathbb{R}P^1$? Is this continuous?

Q2 Atlases:

(i) Suppose our manifold is a circle S^1 embedded in $\mathbb{R}^2$ via the equation $x^2 + y^2 = 1$. Let us define an atlas with two charts with coordinate maps φ_1 and φ_2 via

$$\begin{aligned} \varphi_1^{-1} :\ & (0, 2\pi) \to S^1 \\ \varphi_1^{-1} :\ & \theta \to (\cos(\theta), \sin(\theta)) \end{aligned} \tag{3}$$

and

$$\begin{aligned} \varphi_2^{-1} :\ & (-\pi, \pi) \to S^1 \\ \varphi_2^{-1} :\ & \theta \to (\cos(\theta), \sin(\theta)). \end{aligned} \tag{4}$$

Notice that φ_i^{-1} have different domains and are invertible.

Verify that the maps $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$ are smooth.

(ii) Stereographic coordinates for the sphere work as follows. Imagine the sphere embedded in $\mathbb{R}^3$ via the constraint $x^2 + y^2 + z^2 = 1$. Then we join the North Pole $(0, 0, 1)$ to any other point (x, y, z) on the sphere by a straight line and continue that line until it intersects with the $z = 0$ plane $(X, Y, 0)$. Show that the point of intersection is given by

$$X = \frac{x}{1-z} \quad Y = \frac{y}{1-z}. \tag{5}$$

Mapping points on the sphere to the $z = 0$ plane in this way gives us a coordinate map. Can we use this coordinate map do define a chart covering the entire sphere?

(iii) Suppose we use polar coordinates for the sphere θ and ϕ, such that the points on the sphere are given by

$$x = \sin\theta\cos\phi \quad y = \sin\theta\sin\phi \quad z = \cos\theta. \tag{6}$$

Show that polar and stereographic coordinates are related via

$$X = \cot\frac{\theta}{2}\cos\phi \quad Y = \cot\frac{\theta}{2}\sin\phi. \tag{7}$$

Do polar coordinates suffice for a single chart covering the entire sphere?

Q3 Tangent vectors:

(i) Show that all tangent vectors at a point p in an n dimensional manifold are a vector space.

(ii) Suppose we have the path $\gamma(t) : (a, b) \to \mathcal{M}$, with $(a, b) \subset \mathbb{R}$ and $\gamma(0) = p \in \mathcal{M}$. We also have coordinate functions q^i, such that $q^i(\gamma(t)) = x^i(t)$. Then we have the tangent vector $X = X^i \frac{\partial}{\partial x^i} \in T\mathcal{M}_p$, with

$$X^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}. \tag{8}$$

(Really we should write $x^i(t)$ instead of x^i.)

If t is interpreted as time and $\gamma(t)$ is a particle's position, then how would you interpret $X[x^i(t)]$?