Symplectic Geometry and Classical Mechanics
Exercise Sheet 2: Lagrangians and forms
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Due on 10^{th} November

Q1 Lagrangian dynamics: The action is defined to be

$$S[\gamma] = \int_{t_1}^{t_2} \mathrm{d}t\, L(\gamma(t), \dot{\gamma}(t)), \tag{1}$$

where $\dot{\gamma}(t) = \mathrm{d}\gamma(t)/\mathrm{d}t$. In a chart, with $x^i(t) = \varphi(\gamma(t))$, abusing notation, the Lagrangian is $L(x^i(t), v^i(t))$, where $v^i(t) = \mathrm{d}x^i(t)/\mathrm{d}t$.

(i) [**5 points**] By finding the extremum of the action under variations of the path $x^i(t)$ that keep the endpoints fixed, show that the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial x^i}. \tag{2}$$

(ii) [**5 points**] Suppose that $L = \frac{1}{2} g_{ij}(x_\eta)\dot{x}^i \dot{x}^j$, where $g_{ij}(x_k) = g_{ji}(x_k)$ is a function of position, and $g^{ik} g_{kj} = \delta^i_j$. Show that the equations of motion are

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j \dot{x}^k = 0, \tag{3}$$

where

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}\left(\frac{\partial g_{bd}}{\partial x_c} + \frac{\partial g_{cd}}{\partial x_b} - \frac{\partial g_{bc}}{\partial x_d}\right). \tag{4}$$

Here $g_{ij}(x_k)$ defines a metric on the configuration space.

(iii) [**5 points**] Equation (3) is the geodesic equation. For a free particle in flat space, with $g_{ij}(x_k) = \delta_{ij}$, solve the geodesic equation.

Q2 Forms:

(i) [**5 points**] Given n differential 1-forms ω_i and a permutation π of $\{1, ..., n\}$, show that

$$\omega_{\pi(1)} \wedge \omega_{\pi(2)} ... \wedge \omega_{\pi(n)} = \mathrm{sgn}(\pi)\, \omega_1 \wedge \omega_2 ... \wedge \omega_n. \tag{5}$$

(ii) [**5 points**] Prove that the dimension of the vector space $\bigwedge^k V$ is $\binom{n}{k}$, where n is the dimension of V. What is the dimension of $\bigwedge^n V$?

(iii) [**5 points**] What form do differential 3-forms take on the manifold $\mathcal{M} = \mathbb{R}^3$ with coordinates (x, y, z)? Is this manifold orientable?

Q3 The exterior derivative:

(i) **[5 points]** A form ω is called closed if $\mathrm{d}\omega = 0$. It is called exact if there exists a form η, such that $\omega = \mathrm{d}\eta$. Show that all exact forms are closed.

(ii) **[5 points]** Show that all (closed) differential 1-forms on the manifold $\mathbb{R}$ are exact.

(iii) **[5 points]** It is not true, however, that closed forms are always exact. Suppose we have the following form on $\mathbb{R}^2 - \{(0,0)\}$

$$\omega = \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y. \tag{6}$$

Show that ω is closed.

Why isn't ω exact? (Hint: what is ω on the circle with $x^2 + y^2 = 1$? Use polar coordinates for x and y.)