Symplectic Geometry and Classical Mechanics
Exercise Sheet 3: Integration on Manifolds and more Forms
– Terry Farrelly
Due on 24th November

Q1 Orientation:

(i) [**5 points**] Roughly speaking, a Möbius strip consists of two rectangles glued together at both ends with a twist.

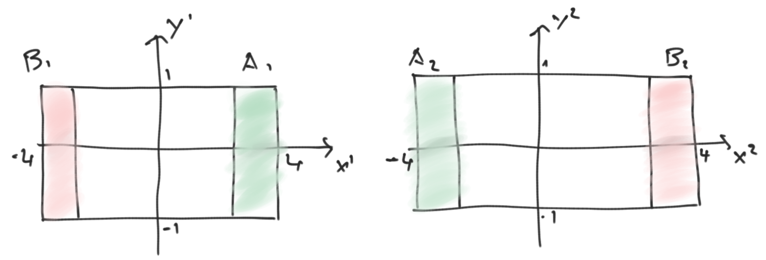

Figure 1: Charts for the Möbius strip. (I'm not an artist!)

Here is an atlas for it. In one chart we have coordinates (x^1, y^1) and in the second we have coordinates (x^2, y^2). Both are in the same range:

$$x^1, x^2 \in (-4, 4) \text{ and } y^1, y^2 \in (-1, 1). \tag{1}$$

In the overlapping regions A_1 and A_2, we have $y^1 = y^2$ and $x^1 = x^2 + 7$. In the other overlapping regions B_1 and B_2, we have $y^1 = -y^2$ and $x^2 = x^1 + 7$.

Explain why this atlas for the Möbius strip is not oriented.

(ii) [**5 points**] Explain why all one-dimensional manifolds are orientable.

Q2 Stokes' theorem: this is a useful tool for integration on manifolds: given an m dimensional oriented manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ and a differential $m-1$ form ω, then

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega. \tag{2}$$

Note: An ($m > 1$ dimensional) manifold with a boundary has both *interior* and *exterior* points. Interior points always have a neighbourhood homeomorphic to the unit open ball in $\mathbb{R}^m$, whereas every boundary point has a neighbourhood homeomorphic to $\{(x_1, ..., x_m) | x_1 \geq 0 \text{ and } \sum_i x_i^2 < 1\}$, with the boundary point itself being mapped to a coordinate with $x_1 = 0$. An example of such a manifold is the closed unit disk in $\mathbb{R}^2$, which has the unit circle as its boundary. (The boundary of an m dimensional manifold is an $m-1$ dimensional manifold.)

(i) **[5 points]** Suppose we have the integral

$$\int_{\mathcal{C}} \frac{1}{2} \left(x \, \mathrm{d}y - y \, \mathrm{d}x \right) , \tag{3}$$

where $\mathcal{C}$ is some closed curve in $\mathbb{R}^2$. What does this tell us about the region enclosed by $\mathcal{C}$?

(ii) **[5 points]** Suppose that $\mathcal{M}$ is embedded in $\mathbb{R}^2$ with coordinates (x, y) satisfying $x \in [-1, 1]$ and $y \in [0, \sqrt{1 - x^2}]$. (In the special case where $\mathcal{M}$ is a submanifold of $\mathbb{R}^2$, Stokes' theorem is known as Green's theorem.) Evaluate the integral

$$\int_{\partial \mathcal{M}} y^2 \mathrm{d}x + 3xy \mathrm{d}y. \tag{4}$$

(iii) **[5 points]** When $\mathcal{M}$ is a region in $\mathbb{R}^3$ with boundary given by a closed surface, Stokes' theorem is sometimes known as the Gauss divergence theorem. Let $\mathcal{M}$ be a submanifold of $\mathbb{R}^3$ that has coordinates (x, y, z) satisfying

$$x \in [0, 1], \; y \in [0, 3], \; z \in [0, 2]. \tag{5}$$

Consider the scary two form

$$F = (3z + x^{10}) \mathrm{d}x \wedge \mathrm{d}y + \left(\frac{1}{2} x^2 + e^{yz^3} \right) \mathrm{d}y \wedge \mathrm{d}z + (3x^x + z - y^2) \mathrm{d}x \wedge \mathrm{d}z. \tag{6}$$

Calculate $\int_{\partial \mathcal{M}} F$.

Q3 Electromagnetism: Let $\mathcal{M}$ be a spacetime manifold with coordinates x^μ, where x^0 is time and the remaining x^i are spatial coordinates. Suppose we have the 1-form $A = A_\mu \mathrm{d}x^\mu$, where A_μ is the electromagnetic vector potential.

(i) **[5 points]** We define the electromagnetic field strength two form by $F = \mathrm{d}A$. Is there a unique A for every F?

(ii) **[5 points]** We have

$$\begin{aligned} F = & B_1 \, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + B_2 \, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + B_3 \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ & + E_1 \, \mathrm{d}x^1 \wedge \mathrm{d}x^0 + E_2 \, \mathrm{d}x^2 \wedge \mathrm{d}x^0 + E_3 \, \mathrm{d}x^3 \wedge \mathrm{d}x^0 , \end{aligned} \tag{7}$$

where E_i and B_i are components of the electric and magnetic field respectively. Show that $\mathrm{d}F = 0$. What does this equation correspond to physically? (Hint: expressions like $\sum_i \partial E_i / \partial x^i = \vec{\nabla} . \vec{E}$ should be familiar from your electromagnetism course.)

(iii) **[5 points]** Another useful operator is the Hodge star operator, which takes k forms on an n dimensional manifold to $n - k$ forms. Here we just need to know how the operator acts on two forms:

$$\begin{aligned}
\star(dx^0 \wedge dx^1) &= -dx^2 \wedge dx^3 \\
\star(dx^0 \wedge dx^2) &= dx^1 \wedge dx^3 \\
\star(dx^0 \wedge dx^3) &= -dx^1 \wedge dx^2 \\
\star(dx^1 \wedge dx^2) &= dx^0 \wedge dx^3 \\
\star(dx^1 \wedge dx^3) &= -dx^0 \wedge dx^2 \\
\star(dx^2 \wedge dx^3) &= dx^0 \wedge dx^1.
\end{aligned} \tag{8}$$

Show that $\star F$ essentially swaps E^i and B^i. Find $\star\star F$.

(iv) **[5 points]** We also have the electric current 3-form:

$$J = \rho\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 - j_1\, \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 - j_2\, \mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 - j_3\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2. \tag{9}$$

F also obeys $\mathrm{d} \star F = J$ (which can be derived as Lagrangian equations of motion). By writing this in terms of E_i and B_i, explain what this equation corresponds to physically.

Show that J obeys the continuity equation $\mathrm{d}J = 0$.