Symplectic Geometry and Classical Mechanics
Exercise Sheet 4: Symplectic forms
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Due on 8^{th} of December

Q1 Symplectomorphisms:

(i) [**5 points**] When is a skew-symmetric bilinear map on a vector space symplectic?

Often we write our symplectic forms as matrices. Evaluating the form on two vectors $\vec{u}$ and $\vec{v}$ gives $\vec{u}^T\Omega\vec{v}$, where T denotes the transpose. In terms of matrices, what does skew-symmetry of a bilinear form look like?

(ii) [**5 points**]

As an example, on $\mathbb{R}^4$ we could take the following symplectic form:

$$\Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{1}$$

Is the transformation on $\mathbb{R}^4$ given below a symplectomorphism? (I.e., does it map Ω to another symplectic form?)

$$N = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \tag{2}$$

(iii) [**Bonus 0 points**] Is there a simple way to check if a transformation given by some matrix N is a symplectomorphism on $\mathbb{R}^{2n}$?

Q2 Symplectic Matrices: Symplectic matrices *preserve* a symplectic form: given a symplectic form J on $\mathbb{R}^{2n}$, a symplectic matrix S satisfies $S^TJS = J$. From here on, we take

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \tag{3}$$

where $\mathbb{1}$ is the $n \times n$ identity matrix.

(i) [**5 points**] Given a symplectic matrix S, show that S and S^{-1} have the same eigenvalues.

(ii) [**5 points**] We know that any symmetric matrix has real eigenvalues and can be diagonalized by an orthogonal transformation. Williamson's theorem is a slightly different result: any positive definite symmetric matrix M on $\mathbb{R}^{2n}$ can be diagonalized by a symplectic transformation in the following way. There exists a symplectic matrix S such that $S^T MS = D$, where

$$D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \tag{4}$$

where Λ is a diagonal matrix whose diagonal elements are the absolute values of the eigenvalues of JM.

Given

$$M = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \tag{5}$$

what is its Williamson normal form, i.e., what is D? (You don't need to find S.)

Q3 Symplectic Group:

(i) [**5 points**] Show that symplectic matrices on $\mathbb{R}^{2n}$ form a group. This group is called $\mathrm{Sp}(n)$ (confusingly, sometimes people write $\mathrm{Sp}(2n)$ or $\mathrm{Sp}(2n, \mathbb{R})$ instead).

(ii) [**5 points**] Take

$$S = \begin{pmatrix} e^{-tA} & 0 \\ 0 & e^{tA} \end{pmatrix}, \tag{6}$$

where $t \in \mathbb{R}$ and A is any symmetric matrix. Show that S is a symplectic matrix. Is the symplectic group compact?

(iii) [**5 points**] Show that $\mathrm{Sp}(1) = SL(2, \mathbb{R})$, which is the set of 2×2 real matrices with determinant one.

Q4 Symplectic Manifolds:

(i) [**5 points**] Show that a $2n$ dimensional symplectic manifold is always orientable. Hint: denoting the symplectic form by Ω, argue that $\Omega^{\wedge n} = \Omega \wedge ... \wedge \Omega$ is a nowhere zero volume form.

Is a symplectic manifold orientable?

(ii) [**5 points**] Going the other way, show that if we have a two form μ on a $2n$ dimensional manifold, and $\mu^{\wedge n}$ is nowhere zero, then μ is a symplectic form.