**Symplectic Geometry and Classical Mechanics**
Exercise Sheet 5: The Simple Pendulum and Hermitian Matrices
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Due on 12th of January

**Q1 Simple Pendulum:** The simple pendulum is composed of a rigid rod of length $l$ that is fixed at one end (so it can rotate) and has a weight of mass $m$ at the other end. We assume that the pendulum can only move in a fixed plane. Also, the only force acting on the pendulum is gravity, pointing downwards, which is constant (set $g$ the acceleration due to gravity to one for simplicity).

(i) [**5 points**] Let $\theta$ be the oriented angle between the rod and the vertical direction. Let $\xi$ be the coordinate along the fibers of $T^*S^1$ induced by the standard angle coordinate on $S^1$. Show that the Hamiltonian function $H : T^*S^1 \to \mathbb{R}$ given by

$$H(\theta, \xi) = \frac{\xi^2}{2ml^2} + ml(1 - \cos(\theta)), \tag{1}$$

is appropriate to describe the single pendulum: check that gravity gives rise to the potential energy $V(\theta) = ml(1 - \cos(\theta))$, and that the kinetic energy is given by $K(\theta, \xi) = \xi^2/(2ml^2)$.

(ii) [**5 points**] Suppose that $m = l = 1$ to make things simpler. Plot the level curves of $H$ in the $(\theta, \xi)$ plane.

(iii) [**5 points**] Show that there exists a number $c$ such that for $0 < h < c$ the level curve $H = h$ is a disjoint union of closed curves.
Show that the projection of each of these curves onto the $\theta$-axis is an interval of length less than $\pi$.
Show that neither assertion is true if $h > c$.
What types of motion are described by these two types of curves? What about when $H = c$?

(iv) [**5 points**] Compute the critical points of the function $H$. Show that, modulo $2\pi$ in $\theta$, there are exactly two critical points: one point $s$ where $H = 0$ and one point $u$ where $H = c$. These points are called *stable* and *unstable* points of $H$ respectively.

(v) [**5 points**] Justify the terminology stable and unstable: show that a trajectory of a Hamiltonian vector field of $H$ with initial point close to $s$ stays close to $s$ forever, and show that this is not true for $u$. What is happening physically?

**Q2 Hermitian matrices:** Let $V$ be the vector space of $n \times n$ complex hermitian matrices. There is an action of the unitary group $\mathrm{U}(n)$ by conjugation, with $\xi \to A\xi A^{-1}$, where $A \in \mathrm{U}(n)$ and $\xi \in V$.
Let $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$, then $V_\lambda$ denotes the subset of $V$ containing matrices whose spectrum is $\lambda$.

(i) **[5 points]** Show that the orbits of the U($n$) action are the manifolds $V_\lambda$.

For a fixed $\lambda \in \mathbb{R}^n$, what is the stabilizer of a point in $V_\lambda$? (Hint: if the $\lambda_i$ are distinct, then the stabilizer of the diagonal matrix is the torus $\mathbb{T}^n$ of all diagonal unitary matrices.)

(ii) **[5 points]** Show that the symmetric bilinear form on $V$, $(X, Y) \to \mathrm{trace}[XY]$ is non-degenerate.

For $\xi \in V$, define a skew-symmetric bilinear form $\omega_\xi$ on $\mathfrak{u}(n) = T_1 U(n) = iV$ (the space of skew-hermitian matrices) by $\omega_\xi(X, Y) = i\mathrm{trace}([X, Y]\xi)$, where $X, Y \in iV$.

Check that $\omega_\xi(X, Y) = i\mathrm{trace}(X(Y\xi - \xi Y))$ and $Y\xi - \xi Y \in V$.

Show that the kernel of $\omega_\xi$ is $K_\xi := \{Y \in \mathfrak{u}(n) | [Y, \xi] = 0\}$.

(iii) **[5 points]** Show that $K_\xi$ is the Lie algebra of the stabilizer of $\xi$. (Hint: differentiate the relation $\xi = A\xi A^{-1}$.)

Show that the $\omega_\xi$s induce nondegenerate two forms on the orbits $V_\xi$.

Show that these two forms are closed.

Conclude that the orbits $V_\xi$ are compact symplectic manifolds.

(iv) **[5 points]** Show that, for any skew-hermitian matrix $X \in \mathfrak{u}(n)$, the vector field on $V$ generated by $X$ for the U($n$)-action by conjugation is $X^\#_\xi = [X, \xi]$.