Symplectic Geometry and Classical Mechanics
Exercise Sheet 6: Bonus problems
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Exercise class on 2nd of February

Q1 Complex manifolds:
Good examples of complex manifolds are complex projective spaces $\mathbb{C}P^n$, which are defined analogously to real projective spaces: $\mathbb{C}P^n$ is the set of lines through the origin in $\mathbb{C}^{n+1}$. More concretely, we say that two points $\vec{z}, \vec{w} \in \mathbb{C}^{n+1}$ define the same line if there is a $\lambda \in \mathbb{C}$ such that $\vec{z} = \lambda \vec{w}$ with $\lambda \neq 0$ and $\vec{z}, \vec{w} \neq 0$.
For each $i \in \{1, ..., n+1\}$, we define U_i to be the set of lines with $z_i \neq 0$. Then we define coordinate maps on each U_i by

$$\begin{aligned} \varphi_i &: \ U_i \to \mathbb{C}^n \\ \varphi_i &: \ (z_1, ..., z_{n+1}) \to \left(\frac{z_1}{z_i}, ... \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, ..., \frac{z_{n+1}}{z_i} \right), \end{aligned} \tag{1}$$

which is well defined because $z_i \neq 0$ when $\vec{z} \in U_i$.

(i) **[Bonus 0 points]** The Riemann sphere can be defined to be $\mathbb{CP}^1$. Instead of using the coordinates above, we can use stereographic coordinates. (Recall stereographic coordinates from exercise sheet 1.)

We embed the sphere in $\mathbb{R}^3$ via the constraint $x^2 + y^2 + z^2 = 1$. Then we join the North Pole $(0, 0, 1)$ to any other point (x, y, z) on the sphere by a straight line and continue that line until it intersects with the $z = 0$ plane at $(X, Y, 0)$. Now let's consider the $z = 0$ plane as the complex plane in the following way: identify $(X, Y, 0)$ with $X + iY$. Therefore, the point (x, y, z) on the Riemann sphere corresponds to the complex number

$$\zeta = \frac{x + iy}{1 - z}. \tag{2}$$

We need a second chart to cover the sphere. We join the South pole $(0, 0, 1)$ to a point (x, y, z) on the sphere to get the complex coordinate

$$\xi = \frac{x - iy}{1 + z}. \tag{3}$$

So we can cover the sphere with two complex charts. What is the relation between the coordinates where the two charts overlap? Is this holomorphic? (What is the relation with the coordinates given above via the coordinate maps φ_i)?

(ii) **[Bonus 0 points]** Complex projective spaces are actually more relevant for quantum mechanics than classical mechanics. Suppose we have a pure quantum state $|\psi\rangle \in \mathbb{C}^n$.[1] If we do a measurement with outcomes $i = \{1, ..., d\}$, then the probability of getting outcome i is

$$\text{prob}(i) = \frac{\langle\psi| M_i |\psi\rangle}{\langle\psi|\psi\rangle}, \tag{4}$$

[1] $|\cdot\rangle$ is quantum mechanical notation for a vector in a complex vector space. $\langle\cdot|$ is the adjoint vector, and $\langle a|b\rangle$ is the complex inner product. People typically reserve $|\cdot\rangle$ for *normalized* vectors. We are not following this convention here.

where M_i are linear operators on $\mathbb{C}^n$.[2] In physics we can only determine the outcome i and these probabilities experimentally. Explain why physical states then should be associated with elements of $\mathbb{C}P^{n-1}$ rather than $\mathbb{C}^n$.

(iii) **[Bonus 0 points]** For $n = 2$ there is a rather useful set of coordinates for pure states in $\mathbb{C}^2$, i.e. $\mathbb{C}P^1$. This is known as the Bloch sphere in quantum mechanics. Let $|0\rangle$ and $|1\rangle$ be a basis for $\mathbb{C}^2$. Show that any point in $\mathbb{C}P^1$ can be identified with

$$\cos(\phi/2)\,|0\rangle + e^{-i\theta}\sin(\phi/2)\,|1\rangle\,. \tag{5}$$

What is the range of ϕ and θ? Are ϕ and θ a chart for $\mathbb{C}P^1$ viewed as a complex manifold?

(iv) **[Bonus 0 points]** Linear complex structures are important for complex manifolds. A real vector space $\mathbb{R}^{2n}$ has a linear complex structure if there is a real matrix J satisfying $J^2 = -\mathbb{1}$. Then we define $(x + iy)\vec{v}$ to be $(x + Jy)\vec{v}$ where $\vec{v} \in \mathbb{R}^{2n}$.

A real linear transformation $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a complex linear transformation of the corresponding complex space $\mathbb{C}^n$ if and only if A commutes with J. Show this for $n = 2$.

Q2 Back to Classical Mechanics:

(i) **[Bonus 0 points]** Gromov's non-squeezing theorem is a powerful result in symplectic geometry. Suppose we have the symplectic spaces, $\mathbb{R}^{2n}$ with coordinates $(x_1, \dots, x_n, p_1, \dots, p_n)$, the ball of radius R

$$B(R) = \{z \in \mathbb{R}^{2n} |\ \|z\| < R\}, \tag{6}$$

and the cylinder of radius r

$$C(r) = \{z \in \mathbb{R}^{2n} | x_1^2 + p_1^2 < r^2\}, \tag{7}$$

each of which has the usual symplectic form

$$\omega = dx_1 \wedge dp_1 + \cdots + dx_n \wedge dp_n. \tag{8}$$

Gromov's non-squeezing theorem states that there is no symplectic embedding of $B(R)$ into $C(r)$ unless $r > R$.

Consider the case with $n = 1$. Suppose you rescale the x_1 coordinate by $\lambda < r/R$, what form must the map take on (x_1, p_1) to preserve the symplectic form?

(ii) **[Bonus 0 points]** Symplectic maps are a special case of volume preserving maps. (A consequence is that Hamiltonian dynamics preserve the volume form, which is known as Liouville's theorem.) For the case $n = 2$, give a volume preserving map that embeds $B(R)$ into $C(r)$. Can you give one for $n = 1$?

[2] These are called POVM elements. They are positive operators and satisfy $\sum_i M_i = \mathbb{1}$. A special case are familiar projective measurements, where each M_i is simply a projector.

(iii) **[Bonus 0 points]** Some particles are freely falling in the earth's gravitational field. Their momenta in the vertical direction are distributed in the range $-b$ to b, and their positions are distributed in the range $-a$ to a. Show that, after a time t, the region in phase space they occupy is still given by $4ab$.

(iv) **[Bonus 0 points]** Prove that the following transformations are canonical

$$P = \frac{1}{2}(p^2 + q^2) \text{ and } Q = \arctan(q/p) \tag{9}$$

$$P = q^{-1} \text{ and } Q = pq^2. \tag{10}$$

(Another name for a canonical transformation is a symplectomorphism.)