Covariant derivative operator \( \nabla \) notation

\[ \nabla \] affine connection

**Notation:** Let \( \mathcal{M} \) be a manifold, denote by \( \mathcal{T}(\mathcal{M}) \) the space of smooth vector fields on \( \mathcal{M} \).

**Definition:** an affine connection \( \nabla \) is a map

\[
\nabla : \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) \rightarrow \mathcal{T}(\mathcal{M})
\]

If immediately extend domain of definition of \( \nabla \) to

\[
\nabla : \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}, \xi) \rightarrow \mathcal{T}(\mathcal{M}, \xi)
\]

(c) Let \( \mathcal{X}, \mathcal{Y} \in \mathcal{T}(\mathcal{M}) \) \( \zeta \in \mathcal{T}(\mathcal{M}, \xi) \):

\[
\nabla (\mathcal{X} + \mathcal{Y}) \zeta = \nabla \mathcal{X} \zeta + \nabla \mathcal{Y} \zeta
\]

(c') \forall \mathcal{X} \in \mathcal{T}(\mathcal{M}) \quad \forall \mathcal{Y} \in \mathcal{T}(\mathcal{M}, \xi) \quad \forall f \in \mathcal{C}(\mathcal{M})

\[
\nabla f \mathcal{X} \mathcal{Y} = f \nabla \mathcal{X} \mathcal{Y}
\]

(i) **Linearity:** \( A, B \in \mathcal{T}(\mathcal{M}, \xi) \):

\[
\nabla \mathcal{X} (A + B) = \nabla \mathcal{X} A + \nabla \mathcal{X} B
\]

**In \( \mathcal{A}(\mathcal{M}) \): note**

\[
\nabla \mathcal{X} = \mathcal{X}^a \nabla \mathcal{A}
\]

\[
\nabla_c (A^a_{\mu \nu} + B^a_{\mu \nu}) = \nabla_c A^a_{\mu \nu} + \nabla_c B^a_{\mu \nu}
\]

2 Leibniz rule: For all \( \mathcal{A} \in \mathcal{T}(\mathcal{M}, \xi) \), \( \mathcal{B} \in \mathcal{T}(\mathcal{M}, \xi') \)

\[
\nabla \mathcal{X} (A \otimes B) = (\nabla \mathcal{X} A) \otimes B + A \otimes (\nabla \mathcal{X} B)
\]

**In \( \mathcal{A}(\mathcal{M}) \):

\[
\nabla_c (A^a_{\mu \nu} + B^a_{\mu \nu}) = (\nabla_c A^a_{\mu \nu} + A^a_{\mu \nu} (\nabla_c B^a_{\mu \nu})
\]
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\[ \nabla_e (A^a_{b_1 \ldots b_k} B^{c_1 \ldots c_l}_{d_1 \ldots d_l}) = (\nabla_e A^a_{b_1 \ldots b_k}) B^{c_1 \ldots c_l}_{d_1 \ldots d_l} + A^a_{b_1 \ldots b_k} (\nabla_e B^{c_1 \ldots c_l}_{d_1 \ldots d_l}) \]

(3) Commutativity with contraction. For all \( A \in \mathcal{F}(\mathbb{N}) \)

\[ \nabla_e \left( \tilde{e}_{j_1 j_2} A \right) = \tilde{e}_{j_1 j_2} \left( \nabla_e A \right) \]

(4) Reduction to vector fields on scalar functions.

Let \( f \in \mathcal{F}(\mathbb{N}) \) then

\[ \nabla_e (f) = X(f) \]

(5) Torsion free:

\[ \nabla_e \nabla_f g - \nabla_f \nabla_e g - [X,e] g = 0 \]

(\text{ex.}) \( \forall f \in \mathcal{F}(\mathbb{N}) \)

\[ \nabla_e \nabla_f g = \nabla_f \nabla_e g \]

Simple consequences:

\[ \forall X \in \mathfrak{X}(\mathbb{N}) \quad Y, Z \in \mathfrak{X}(\mathbb{N}) \quad f \in \mathcal{F}(\mathbb{N}) \]

(6) \[ \nabla_e (f Y) = X(f) Y + f \nabla_e Y \]

(\text{ex.}) \[ [\nu, \omega] (f) = \nu(\omega(f)) - \omega(\nu(f)) \]

\[ = \nu^a \nabla_\nu (\omega^b \nabla_b (f)) - \omega^b \nabla_\omega (\nu^a \nabla_a (f)) \]

\[ = \nu^a (\nabla_\nu \omega^b) \nabla_b f - \omega^b (\nabla_\omega \nu^a) \nabla_a f \]

\[ + \nu^a \omega^b \nabla_{\nu a} \nabla_{\omega b} f - \nu^a \omega^b \nabla_{\omega b} \nabla_{\nu a} f \]

So vector field \([\nu, \omega] \) is given by

\[ [\nu, \omega]^b = \nu^a \nabla_a \omega^b - \omega^a \nabla_a \nu^b \]

There are many derivative operators obeying (3), (4), (5).

The simplest example is the ordinary derivative operator.
A defined as follows. Let $\psi$ be a chart & $T \in \mathcal{J}(\psi)$ with components $T^a_{\nu_1 \nu_2}$ in coord. basis.

As a map on $f(\mathcal{J}(\psi))$,

\[ (\tilde{\nabla}_a - \nabla_a)(f) = \tilde{\nabla}_a f - \nabla_a f = 0 \]

Now consider

\[ (\tilde{\nabla}_a - \nabla_a) \text{ on } \mathcal{J}(0,1) \]

\[ (\tilde{\nabla}_a - \nabla_a) : \mathcal{J}(0,1) \rightarrow \mathcal{J}(0,2) \]

Let $f \in \mathcal{J}(0,1)$ & $u \in \mathcal{J}(0,1)$. Then $fu_a \in \mathcal{J}(0,1)$

We find

\[ (\tilde{\nabla}_a - \nabla_a)(fu_a) = f(\tilde{\nabla}_a - \nabla_a)(u_a) \]

Claim: (k) implies that $\tilde{\nabla}_a u_b - \nabla_a u_b$ only depends on $u_b$ at $p$. Strategy:

Let $u'_b \in \mathcal{J}(0,1)$ have property

\[ u'_b \Big|_p = u_b \Big|_p \]

We will argue:

\[ \tilde{\nabla}_a u_b - \nabla_a u_b = \tilde{\nabla}_a u'_b - \nabla_a u'_b \]

One can find $(\omega')_b$ smooth functions for which vanish at $p$ and smooth $\mu'_b$ such that

\[ \omega'_b - \omega_b = \sum_{\alpha=1}^{2} f_{\alpha} \mu'_b(\alpha) \]

Apply

\[ (\tilde{\nabla}_a - \nabla_a)(\omega'_b - \omega_b) = \sum_{\alpha=1}^{2} f_{\alpha} (\tilde{\nabla}_a - \nabla_a) \mu'_b(\alpha) \]
Hence 
\( (\nabla_a - \nabla_a) \omega_b = \tilde{\nabla}_{a-b} \omega_b \) \( a \neq b \) 

doesn't depend on \( \omega_b \) away from \( a \) 

\( (\nabla_a - \nabla_a) \) : as a map of \( \text{tensor of type } (0,1) \) to \( \mathcal{F} (a, b) \) 

\( C : V^* \rightarrow V^* \otimes V^* \)

By duality \( V^* = V^* \), \( C \) can be interpreted as a map from \( V^* \otimes V^* \otimes V^* \rightarrow \mathbb{R} \) as an element of \( V^* \otimes V^* \otimes V^* \). We write \( C = c^a_{bc} \) for the \( \text{antisymmetric } c^a_{bc} \)

\( \nabla \omega_b = \tilde{\nabla}_{a-b} \omega_b - c^a_{bc} \omega_c \)

Consider \( \omega_b = \vec{\nabla} f = \tilde{\nabla} f + f \cdot \vec{\nabla} f \)

\( \Delta \vec{\nabla} f = \tilde{\nabla}_{a-b} \vec{\nabla} f - c^a_{bc} \vec{\nabla} f \)

\( \text{symmetry} \quad \text{symmetry} \)

\( c^a_{bc} = c^c_{ba} \)

The Levi-Civita property \( (4) \) was determined 
\( (\nabla_a - \nabla_a) \) on all \( \text{tensor fields} \): let \( \omega \in \mathcal{F} (a, b) \)
and \( t \in \gamma (1,0) = \mathcal{F} (n) \)

\( \omega_a t^a \in \mathcal{F} (n) \)

(41):
\( (\nabla_a - \nabla_a) (\omega_b t^b) = 0 \)

Levi-Civita (4):
\( t^c (\nabla_a - \nabla_a) \omega_b + \omega_b (\nabla_a - \nabla_a) t^c = 0 \)

\( \Rightarrow \) \( t^c c^c_{ab} \omega_c + \omega_b (\nabla_a - \nabla_a) t^c = 0 \)

\( \Rightarrow \omega_b (\nabla_a - \nabla_a) t^c + t^c c^c_{ab} = 0 \)

or

\( \nabla_a t^b = \tilde{\nabla}_a t^b + c^c_{ab} t^c \)
In other words,\( \tilde{\nabla}_a \) is determined by \( C^c_{ab} \equiv \left[ V_a \otimes V_b \otimes V_c^* \right] \).

Conversely, \( C^c_{ab} \equiv V_a \otimes V_b \otimes V_c^* \) and a (\( B, C, C \)) derivative \( \tilde{\nabla}_a \) in \( \tilde{\nabla}_a \) defined by (94) is also a (\( B, C, C \)) derivative.

\( C^c_{ab} \) doesn't always transform as a tensor field of type \( (1,2) \) under change of coordinates (e.g., when \( \tilde{\nabla}_a = \partial_a \)).

Most important example: let \( \tilde{\nabla}_a = \partial_a \). In this case \( C^c_{ab} \rightarrow \Gamma^c_{ab} \), called Christoffel symbols.

\begin{align*}
\partial_a t^b &= \tilde{\nabla}_a t^b + \Gamma^b_{ac} t^c \\
\tilde{\nabla}_a t^\nu &= \partial_a t^\nu + \Gamma^\nu_{a\rho} t^\rho
\end{align*}