We discuss curved continua \textit{microscopically}.

\[ \Rightarrow \text{Manifolds} \]

\textit{Notation:} \[ \mathbb{R}^n = \{ (x_1, \ldots, x^n) | x_j \in \mathbb{R} \} \]

\[ x = (x_1, \ldots, x^n) \]

\[ |x-y| = \left( \sum_{x=1}^{n} (x^x-y^x)^2 \right)^{\frac{1}{2}} \]

\[ B_r(y) = \{ x \in \mathbb{R}^n | |x-y| < r \} \]

\textit{open ball} around \( y \)

\textit{Open set} \( U \):

\[ \forall x \in U \quad \exists \varepsilon > 0 \quad s.t. \quad B_\varepsilon(x) \subset U \]

\[ C^0 = \text{set of } n \text{-times differentiable functions} \]

\[ C^\infty = \text{infinitely differentiable functions} \]

\[ C^\infty = \text{smooth} \]

\[ \text{We do things in n dimensions} \]
Definition. An \( n \)-dimensional, \( C^\infty \), real manifold \( M \) is a set together with a collection \( \{ \mathcal{O}_\alpha \} \) of subsets satisfying:

1. \( M \) is a topological space, Hausdorff & second countable.
2. \( \{ \mathcal{O}_\alpha \} \) are homeomorphisms.
3. The set \( \mathcal{O}_\alpha \) covers \( M \), \( \cup \mathcal{O}_\alpha = M \). Or \( \forall \alpha \in \Omega \) at least one \( \alpha \) s.t. \( p \in \mathcal{O}_\alpha \).

For each \( \alpha \) there is a 1-1 and onto map \( \Psi_\alpha : \mathcal{O}_\alpha \to U_\alpha \), where \( U_\alpha \subset \mathbb{R}^n \).

If any \( \mathcal{O}_\alpha \) and \( \mathcal{O}_\beta \) overlap, \( \Psi_\alpha \circ \Psi_\beta^{-1} \), which acts via:

\[
\Psi_\beta \circ \Psi_\beta^{-1} : \Psi_\beta [\mathcal{O}_\beta \cap \mathcal{O}_\alpha] \to \Psi_\beta [\mathcal{O}_\beta \cap \mathcal{O}_\alpha] \\
U_\beta \subset U_\alpha \subset \mathbb{R}^n
\]

is \( C^\infty \)
The maps $\psi_\alpha$ are called **charts** (mathematics) or **coordinate systems** (physics).

The definition so far depends on $\{U_\alpha, \psi_\alpha\}$: if we add a new set $\pi$ to new chart $\psi' \Rightarrow$ new manifold even $U, \pi \& \psi'$ contain no new info! To eliminate this arbitrariness, we require that the cover $\{U_\alpha\}$ be a chart family $\pi U$ maximal. That is, all coordinate systems compatible with (2) $\&$ (3) are required included.

**Examples:**

1. $\mathbb{R}^n$ is a trivial example, can be covered by a single chart $U_\alpha = \mathbb{R}^n$, $\psi = \text{id}$ (identity, $\mathbb{R}^n$ as a manifold has uncountably many covers).

2. Spacetime $\mathbb{R}^1 \times \mathbb{R}^3 \cong \mathbb{R}^{4}$ (Lorentzian structure comes later).

3. The sphere $S^n$ (embedded in $\mathbb{R}^{n+1}$ dimensions)

$$S^n = \{ (x^1, \ldots, x^{n+1}) \mid \sum (x^{n+1})^2 = 1, \ 3 \leq \mathbb{R}^{n+1} \}$$
\[ S^n = \{ (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{m=1}^{n+1} (x^m)^2 = 1 \} \subset \mathbb{R}^{n+1} \]

Let \[ O^+_x = \{ (x^1, x^2, \ldots, x^n) \in S^n \mid x^n > 0 \} \]

\[ O^-_x = \{ (x^1, x^2, \ldots, x^n) \in S^n \mid x^n < 0 \} \]

There are \( 2(n+1) \) such sets \( \Rightarrow \) cover \( S^n \).

Need coordinate systems

\[ \Psi^+_x : O^+_x \to \mathbb{R}^n \quad \text{and} \quad \Psi^-_x : O^-_x \to \mathbb{R}^n \]

via

\[ \Psi^+_x (x^1, \ldots, x^n) = (x^1, x^2, x^3, \ldots, x^n) \in \mathbb{R}^n \]

Special case \( S^2 \):

\[ (\Psi^+_y \circ (\Psi^+_x)^{-1}) (y, z) = (\sqrt{1-y^2-z^2}, z) \]

\( \Psi \) transition function goes from \( U_x \to U_y \).

Ex. find the rest of these transition functions \( \xi \) and show they are all \( C^\infty \).

Comment: do we need all this manifold stuff anyway?

Yes: Cosmology, Black holes
New manifolds from old

Suppose $M$ and $M'$ are manifolds of dimension $n$ and $n'$ (respectively). We can define product space $M \times M'$.

Suppose $\psi : O \rightarrow U$ and $\psi' : O' \rightarrow U'$ are charts for $M$ and $M'$ (respectively). We define for $M \times M'$ with

$$\psi \times \psi' : O \times O' \rightarrow U \times U' \subset R^{n+n'}$$

with

$$O \times O' = O \times O'$$

$$U \times U' = U \times U' \subset R^{n+n'}$$

and

$$\psi \times \psi'(p, p') = (\psi(p), \psi'(p')) \subset R^{n+n'}$$

Ex. prove this is a manifold

Ex. realize $R^n = R^1 \times R^1 \times \ldots \times R^1$ in two ways

With just $R$ & $S^n$ and their products, one can build many relevant manifolds for GR

Torus: $S^1 \times S^1$ etc etc

Endow class of manifolds with category structure by describing morphisms between them

Let $M$ and $M'$ be manifolds and let $\psi_\alpha : U_\alpha \rightarrow V_\alpha$ and $\psi'_\beta : U'_\beta \rightarrow V'_\beta$ be their charts. A map

$$f : M \rightarrow M'$$

is said to be smooth if $\forall \alpha, \beta$ the map

$$\psi'_\beta \circ f \circ (\psi_\alpha)^{-1} : U_\alpha \rightarrow U'_\beta$$

is a smooth map.
When are two manifolds the same?

If \( f : M \to M' \) is \( C^\infty \), one-to-one, and onto, and has \( C^\infty \) inverse, then it is called a diffeomorphism.

The manifolds \( M \) and \( M' \) are said to be diffeomorphic.

The manifolds \( M \) and \( M' \) have identical manifold structure.

Vectors

Euclidean (or Minkowskian) space has a natural vector space structure:

\[
V = \mathbb{R}^n
\]

is both a manifold & vector space (globally). For general manifolds one loses the vector space structure (no additive structure).

Example

\( S^2 \) is not a vector space.

It is, however, still possible to associate vector space to (embedded) manifolds.

\( \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\)
Problem: how to define things intrinsically?!

Answer: first identify vector space of tangent vectors with vector space of directional derivatives, then we define derivatives intrinsically.

**Definition:** A locally Euclidean space $M$ of dimension $d$ is a Hausdorff topological space $M$ for which each point has a neighborhood homeomorphic to open subset of $\mathbb{R}^d$.

**Definition:** A differentiable structure $\mathcal{F}$ of class $C^k$ ($1 \leq k \leq \infty$) on a locally Euclidean space $M$ is a collection coordinate systems $\{ (\Omega^\alpha, \eta^\alpha) \mid \alpha \in \Lambda \}$ s.t.

(a) $\bigcup_{\alpha \in \Lambda} \Omega^\alpha = M$

(b) $\eta^\alpha \circ \eta^{-1} \in C^k$ for all $\alpha, \beta \in \Lambda$

(c) The collection $\mathcal{F}$ is maximal w.r.t. (b)

A $d$-dimensional differentiable manifold of class $C^k$ ($C^k$ can generalized to $C^{\infty}$, complex, etc.) is a pair $(M, \mathcal{F})$ of a $d$-dimensional, second countable, locally Euclidean space $M$ together with a differentiable structure $\mathcal{F}$ of class $C^k$. 

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