Directional derivative as vectors

Let \((v^1, v^2, \ldots, v^n) \in \mathbb{R}^n\). This defines a directional derivative at a point \(p \in \mathbb{R}^n\) as follows. Suppose \(f\) is a function from \(\mathbb{R}^n \to \mathbb{R}\) and define

\[
\nabla f(p) = \frac{\partial}{\partial x^i} f(x^i) \bigg|_{x^i = p^i}
\]

Conversely, given \((v^1, v^2, \ldots, v^n)\), we obtain a vector \(w \in \mathbb{R}^n\).

Further, directional derivatives at \(p\) form a vector space.

Note: \((v^1, v^2, \ldots, v^n)\) denotes

1. \(\nabla (af + bg) = a \nabla f + b \nabla g\) \(\forall a, b \in \mathbb{R}\)
2. \(\nabla (fg) = f(p) \nabla g + g(p) \nabla f\) (Leibniz property)

Notation: \(\mathcal{F}(\mathbb{R})\) set of all \(C^\infty\) functions from \(\mathbb{R}^n \to \mathbb{R}\)

Idea: define a vector space \(V_p\) associated to point \(p\) to be set of all maps \(\nabla: \mathcal{F}(\mathbb{R}) \to \mathbb{R}\) which obey (1) & (2).

\(\mathbb{R}_x\)

Ex. (a) Convince yourself if \(h \in \mathcal{F}(\mathbb{R})\) is constant then \(\nabla h = 0\), using only (1) & (2).

(b) Prove \(V_p\) is a vector space.

Have we created a monster?! \(\dim(V_p) = \infty\) !

Mean: let \(M\) be an \(n\)-dimensional (smooth) manifold. Let \(p \in M\) and let \(V_p\) denote “tangent space at \(p\)” above.

Then \(\dim(V_p) = n\)

Notation/definition.
Let \( \psi : \mathcal{O} \to U \subset \mathbb{R}^n \) be a chart with \( p \in \mathcal{O} \),
if \( f \in \mathcal{F}(\mathcal{O}) \) then \( f \circ \psi : U \subset \mathbb{R}^n \to \mathbb{R} \) is \( C^\infty \).

Define for \( i = 1, \ldots, n \)
\[
\chi_i (f) = \left( \frac{\partial}{\partial x_i} (f \circ \psi) \right) |_{\psi(p)}
\]
where \( (x_1, \ldots, x^n) \) are the coordinates of \( \mathbb{R}^n \).

Ex. \( \chi_i \) so defined are tangent vectors (i.e., elements of \( \mathfrak{v}_b \)).

Proof: Now suppose \( F : \mathbb{R}^n \to \mathbb{R} \) is \( C^\infty \). Then for all \( a = (a_1, \ldots, a^n) \) \( \exists \) \( C^\infty \) functions \( H_a : \mathbb{R}^n \to \mathbb{R} \) s.t. \( \forall x \in \mathbb{R}^n \)
\[
F(x) = F(a) + \sum_{a_i = 1}^n (x^n - a^n) H_{a_i}(x)
\]
with \( H_{a_i}(a) = \frac{\partial}{\partial x_i} |_{x = a} \).

(Hint: See bonus problem: prove the statement.)

Let \( F(x) = (f \circ \psi)(x) \) and \( a = \psi(p) \). Then by (6.7)
we have for all \( q \in \mathcal{O} \)
\[
f(q) = f(p) + \sum_{\mu = 1}^n \left( (x^n \circ \psi)(q) - (x^n \circ \psi)(p) \right) H_{a_\mu}(\psi(q))
\]
Suppose \( v \in \mathfrak{v}_\mu \). Apply \( \sigma \) to \( f \)
\[
\sigma(v) = \mathcal{V}(f(p) + \sum_{\mu = 1}^n \left( (x^n \circ \psi)(q) - (x^n \circ \psi)(p) \right) H_{a_\mu}(\psi(q)))
\]
\[
\mathcal{V}(H_{a_\mu}(\psi(p)))
\]
\[
\mathcal{V}(\nabla_x x^\mu \psi - x^\mu \nabla_x \psi) \]

Introduction to general relativity Page 2
\[ v = \sum_{\mu} v^\mu X^\mu = \sum_{\mu} v^\mu \left( \frac{\partial \psi^1}{\partial x^\mu} \right) \psi(\mathbf{r}) \]

\[ \Rightarrow \quad v(f) = \sum_{\mu} v^\mu X^\mu(f) \]

The basis \( \{X_\mu | \mu = 1, \ldots, n\} \) of \( V_\mathbf{r} \) is called the coordinate basis, often denoted

\[ X_\mu = \frac{\partial}{\partial x^\mu} \left. \right|_{\mathbf{r}} \]

Suppose we had chosen a different chart \( \psi' \). We would have coord. basis \( \{X'_\nu \} \).

Chain rule (over)

\[ X^\mu = \sum_{\nu} \frac{\partial x^\mu}{\partial x'^\nu} \left. \right|_{\psi} \]

where \( x'^\nu \) denotes \( \nu \)-th component of \( \psi' \circ \psi^{-1} \)

Given a tangent vector in basis \( X_\mu \):

\[ v = \sum_{\mu} v^\mu X^\mu = \sum_{\nu} v'_{\nu} \left( \frac{\partial \psi^1}{\partial x'^\nu} \right) \psi(\mathbf{r}) \]

\[ \Rightarrow \quad v'_{\nu} = \sum_{\mu} v^\mu \left( \frac{\partial \psi^1}{\partial x'^\nu} \right) \]

Vector transformation law.
Vector transformation law.

**Definition:** A smooth curve \( C \) on a manifold \( M \) is a \( C^\infty \) map \( C : \mathbb{R} \to M \) (or a viewed \( f \in C \)).

\[
C : t \to \gamma(t)
\]

To each \( p \in M \) on \( C \) we associate a tangent vector \( T(\gamma) \) as follow: set \( f(t) = \gamma(t) \)

\[
T(f) = \left. \frac{d}{dt} (f \circ C) \right|_t = \sum_{\mu} 2 \frac{d}{dt} (f \circ \gamma) (X^\mu) \left( \frac{dx^\mu}{dt} \right) = \sum_{\mu} \frac{dx^\mu}{dt} X^\mu (f)
\]

where \( x^\mu \cdot (f \circ C) = x^\mu (t) \).

This expansion works for any coord. basis. Components of \( T^\mu = \frac{dx^\mu}{dt} \).

We call \( V_p \) the tangent space at \( p \).

\[ \bigcup_{p \in M} V_p = TM \text{ tangent space for } M. \]

**Warning:** although \( \dim(V_p) = \dim(V_q) \), \( \forall p,q \in M \), and \( V_p \cong V_q \), these isomorphisms are not natural. No standard way to choose two isomorphism \( \Rightarrow \) Isomorphism could be confused. To get “good” choice need extra data!

**Definition:** a tangent field \( \tau \) on a manifold \( M \) is an assignment of a tangent vector \( \tau_p \in V_p, \forall p \in M \). We say \( \tau \) is smooth if \( \forall f \in \mathcal{F}(M), \tau(f) \) is a \( C^\infty \) function.

**Lemma:** The coord. basis fields \( X^\mu \) are smooth.

Proof: \( X^\mu (f)(p) = \sum_{\mu} 2 (f \circ \gamma) (X^\nu) \left. \frac{dx^\nu}{dt} \right|_{\gamma(p)} \) is \( C^\infty \) function.

Since an arbitrary tangent vector \( \tau \) is a linear combination of \( X^\mu : \tau \text{ smooth } \Rightarrow \text{ its components } \tau^\mu \in \mathcal{F}(M) \).
Velocity field \( \nu \) is tangent vector field

Solution to equations of motion is a smooth curve \( C \)

\[ T(f) = \nu(f) \]