Let \( p \in M \) be a point in a manifold \( M \). \( V_p \): tangent space at \( p \). Study behavior of \( v^i \in V_p \), \( \epsilon^i \in V^*_p \) etc. under change of coordinates of \( M \).

1. Dual of \( V_p \): \( V^*_p \): cotangent space; elements of \( V^*_p \) are called covariant vectors. Given basis

\[
e^i = \frac{\partial}{\partial x^i} \quad \text{formally define dual basis}
\]

\[
e^i(v_j) = \delta^i_j
\]

\[
\Rightarrow \quad dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j \quad \Box
\]

\( dx^i \): linear function of tangent vectors defined by \( \Box \)

**Change of coordinate system:**

\[
u'^i = \sum_{r=1}^n v^r \frac{\partial x'^i}{\partial x^r}
\]

(\(\text{Chw transformation law} \))

Let \( \omega \in V^*_p \)

\[
\omega = \sum_{r=1}^n \omega^r \frac{\partial}{\partial x^r}
\]

Apply \( \omega \) to \( v \).

\[
\omega(v) = \omega \left( \sum_{r=1}^n v^r \frac{\partial}{\partial x^r} \right) = \sum_{r=1}^n v^r \omega^r \left( \frac{\partial}{\partial x^r} \right)
\]

\[
= \sum_{r=1}^n \omega(v^r) \frac{\partial}{\partial x^r}
\]

\[
= \sum_{r=1}^n \omega^r \frac{\partial}{\partial x^r}
\]

\[\omega^r = \omega^r \frac{\partial}{\partial x^r} \quad \text{covariant vector transformation law} \]

In general, for a tensor \( T \in \mathcal{S}(r,l) \),
\[ T = \sum_{\nu} T^\nu_\mu \left( \nu \right)_{\mu} \otimes (v^\nu)^* \]

\[ \nu \mu = (\nu_1 \ldots \nu_k) \quad \gamma = (v_1 \ldots v_k) \]

\[ v^\nu = \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^2} \otimes \ldots \otimes \frac{\partial}{\partial x^n} \]

\[ (v^\nu)^* = dx^1 \otimes dx^2 \otimes \ldots \otimes dx^n \]

Components of \( T \) in new coord system: \( \frac{\partial}{\partial x^1} \)

\[ T^{\nu'}_{\mu'} = \sum_{\nu \mu} T^{\nu \mu} \left( \nu' \right)_{\mu'} \otimes (v^\nu)^* \]

\[ \Rightarrow \text{Tensor transformation law} \]

A collection of numbers \( T^\nu_\mu \) transforming like \( \otimes \) is classically called a tensor (field).

A smooth tensor field \( T \) of type \((\nu, \mu)\) is one for which

\[ T(\omega_1 \ldots \omega_k; v_1 \ldots v_\mu) \text{ is smooth for all} \]

\[ \omega \in \Gamma \left( \mathfrak{R}^k \right) \text{ is smooth if} \]

\[ \omega \text{ smooth} \]

Examples: (i) Currents and densities in SR.

Let \( M = \mathbb{R}^{1,3} \).

\[ \delta j(N) \text{ & charge } q_j \]

\( N \) particles: \( j = 1, \ldots, N \)

Density (of charge) \( \Sigma (x, t) = \sum_{j=1}^{N} q_j \delta^{(5)}(x - x_j(t)) \) (Dirac delta function)

Current

\[ j(x, t) = \sum_{j=1}^{N} q_j \delta^{(5)}(x - x_j(t)) \frac{dx_j(t)}{dt} \]

Define a four vector \( J^\mu \) by setting

\[ J = \begin{pmatrix} \mathbf{E} \\ \mathbf{0} \end{pmatrix} \]

Ex. argue that \( J^\mu \) is a vector \((n, V_n)\) field under change of coordinates via Lorentz transformations.
of coordinates via Lorentz transformations

\[ x'^{j} = \Lambda^{j}_{\ i} x^{i} \]

(ii) Energy-momentum tensor \( T_{\mu}^{\nu} \). Let \( M = \mathbb{R}^{1,3} \). Consider a collection \( N \) particles with energy-momentum four vectors

\[ p_{j}^{\nu} \quad j = 1, \ldots, N \]

The density of \( \mu \)-component \( p_{\mu}^{\nu}(x) \) is defined to be

\[ T_{\mu}^{\nu}(x) = \sum_{j=1}^{N} p_{j}^{\mu}(x) \delta^{(4)}(x-x_{j}(t)) \]

Correspondingly, correct

\[ T_{\mu}^{\nu}(x) = \sum_{j=1}^{N} p_{j}^{\mu}(x) \frac{dx_{j}^{\nu}}{dt} \delta^{(4)}(x-x_{j}(t)) \]

Combine to a single formula

\[ T_{\mu}^{\nu}(x) = \sum_{j=1}^{N} p_{j}^{\mu}(x) \frac{dx_{j}^{\nu}(t)}{dt} \delta^{(4)}(x-x_{j}(t)) \]

(Here \( x^{0} = t \)) Since

\[ p_{j}^{0} = E_{j} \frac{dx_{j}^{0}}{dt} \]

we have

\[ T_{\mu}^{\nu}(x) = \sum_{j=1}^{N} \left( p_{j}^{\mu} \frac{dx_{j}^{\nu}}{dt} - p_{j}^{\nu} \frac{dx_{j}^{\mu}}{dt} \right) \delta^{(4)}(x-x_{j}(t)) \]

\[ \Rightarrow \quad T_{\mu}^{\nu} \text{ is symmetric ; i.e. } T_{\mu}^{\nu} = T_{\nu}^{\mu} \]

Writing

\[ T_{\mu}^{\nu}(x) = \int dx^{4} \sum_{j=1}^{N} p_{j}^{\mu} \frac{dx_{j}^{\nu}}{dt} \delta^{(4)}(x-x_{j}(t)) \]

argue (ex), under Lorentz transforms \( \xi^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} \)

\[ T_{\mu}^{\nu} = \Lambda^{\mu}_{\ \alpha} \Lambda^{\nu}_{\ \beta} T_{\alpha \beta} \]

\( \left( \Lambda^{\mu}_{\ \alpha} = \frac{\partial x_{\mu}}{\partial x^{\alpha}} \right) \)

\( T \) is a tensor of type \((2,0)\).

(iii) The metric tensor

A metric tensor \( g \) is a one-field of type \((0,2)\), which is symmetric and non-degenerate, i.e.,

\[ g(v_{i}, v_{j}) = g(v_{j}, v_{i}) \]

and

\[ g(v_{i + 1}, v_{i - 1}) = g(v_{i}, v_{i}) \]
\[ g(u, v) = 0 \quad \forall u \neq v, \quad v_i = 0 \]

A metric is the extra data we need to supply us with a notion of infinitesimal length:

\[ \text{infinitesimal displacement} \quad \Rightarrow \quad \text{tangent vector} \]

\[ \text{"infinitesimal squared distance"} \quad \Rightarrow \quad \text{quadratic function of tangent vector} \]

Choose coordinate basis \( \frac{\partial}{\partial x^i} \): expand \( g \).

\[ g = \sum_{\mu \nu} g_{\mu \nu} \, dx^\mu \otimes dx^\nu \quad (= ds^2) \]

We often omit \( \otimes \) sign:

\[ ds^2 = g = \sum_{\mu \nu} g_{\mu \nu} \, dx^\mu \, dx^\nu \]

A metric actually supplies us with extra data of a binary product on \( V_p \), \( V_p \) on:

\[ (v, w)_p = \sum_{\mu \nu} g_{\mu \nu} \left( dx^\mu \otimes dx^\nu \right) (v, w) \]

\[ = \sum_{\mu \nu} g_{\mu \nu} \left( dx^\mu (v) \, dx^\nu (w) \right) \]

\[ = \sum_{\mu \nu} g_{\mu \nu} \, v^\mu \, w^\nu \]

\[ \Rightarrow \quad v = \sum \frac{v^\mu}{\partial x^\mu} \]

From Schmidt procedure: orthonormal basis \( v_i \) for \( V_p \).

\[ (v_\mu, v_\nu) = g(v_\mu, v_\nu) = \delta_\mu^\nu \]

where \( \delta_\mu^\nu = \delta_{\pm 1} \)

\[ \Rightarrow \quad \text{prove this} \]

The number of \( \pm 1 \) is independent of ortho. basis –> signature of \( g \)

\[ A \text{ metric } g \text{ with } S \neq \pm 1 \text{, } V_m \text{ is Riemannian} \]

\( g \) is positive definite. The metric of spacetime has signature \( (-1, +1, +1, +1) \)

\[ A \text{ metric } g \text{ is simultaneously interpreted as a } (0, 2) \]

\[ \text{tensor and also as a multilinear map from } V_p \times V_p \rightarrow R \]

\[ \text{also as a linear map from } V_0 \rightarrow V^*, \text{ or } V \rightarrow g(\cdot, v) \]
also as a linear map from \( V_p \) to \( V_p^* \) induced via 

\[
    v \mapsto g(\cdot, v) = \omega
\]

What is this? 

\[
    g(\cdot, v) : V_p \to \mathbb{R} \\
    v \in V_p^* 
\]

This map is 1 to 1 & onto and gives us a canonical basis-independent correspondence between vectors & dual vectors.

**Abstract index notation**

Suppose \( T \in \mathcal{S}^{(k,l)} \).

Think \( T \) a multilinear map, for \( (V_p^*)^k \otimes V_l \to \mathbb{R} \).

Can specify \( T \) via its components in a basis \( T^{\alpha_1 \cdots \alpha_k}_{\beta_1 \cdots \beta_l} \).

Often it is enough just to know which arguments of \( T \) take vectors & or dual vectors. Capture this by labelling each argument with a lower case latin letter.

Superscript indices label contravariant indices &

Subscript indices - a covariant index, eg.

\[
    T^{ab}_{cd}
\]

denotes a \((2,2)\) tensor. Here lower case latin letters label arguments & their type (not component with respect to...)

\( \alpha, \beta \)

"Basic"