Introduction to general relativity: abstract index notation; curvature

03 May 2021  08:15

\[ T \in J(k,l) \] (at least) two ways to think about

(i) \[ T \in V^*_p \otimes V^*_p \otimes \cdots \otimes V^*_p \otimes V^*_p \otimes \cdots V^*_p \] (a vector in a vector space)

(ii) \[ T : V^*_p \otimes \cdots \otimes V^*_p \otimes V^*_p \otimes \cdots V^*_p \to \mathbb{R} \] (is a linear map from a vector space to \( \mathbb{R} \))

In way (ii) we think \( T \) as a function with \( k \) indices/arguments from \( \otimes_k V^*_p \) and \( l \) indices/arguments from \( \otimes_l V^*_p \).

\[ T(\cdot, \cdot, \cdots, \cdot, \cdot, \cdots, \cdot) \in \mathbb{R} \]

Evaluation

\[ T(w, u_2, \cdots, u_k, v_2, \cdots, v_l) \]
\[ \in V^*_p \]
\[ \in V^*_p \]

A TN: a way to specify the type of a tensor by naming these entries.

\[ T(\cdot, \cdot, \cdots, \cdot, \cdot, \cdots, \cdot) \to \mathbb{R} \]

To specify the type as follows

\[ T_{abc} \] → not components, they label entries

\[ \text{def} \]
This says: a tensor which takes 3 arguments from $V^*$ & 3 arguments from $V$.

The actual components of $T$ with respect to a coordinate system $x^1, x^2$ are

$$
\sum_{\nu_1, \nu_2, \nu_3} \frac{\partial}{\partial x^{\nu_1}} \frac{\partial}{\partial x^{\nu_2}} \frac{\partial}{\partial x^{\nu_3}} T_{\nu_1 \nu_2 \nu_3}^{\nu_1 \nu_2 \nu_3} dx^{\nu_1} \otimes dx^{\nu_2} \otimes dx^{\nu_3}
$$

for each choice of $\nu_1, \nu_2, \nu_3$, $\nu_1, \nu_2, \nu_3$ this is a function of $M$.

Certain tensor operations can be expressed compactly using

**Aini:**

**Contraction:** if $T_{a_1 \ldots a_k}^{b_1 \ldots b_l} \in \mathcal{S}(k, l)$

then

$$
\delta_{ij} \cdot T = T_{a_1 \ldots a_k}^{b_1 \ldots b_l} \quad \text{if } c \text{ is in the } i \text{th entry}
$$

$$
\delta_{ij} \cdot T = T_{b_1 \ldots b_l}^{a_1 \ldots a_k} \quad \text{if } c \text{ is in the } j \text{th entry}
$$

What type of tensor do we have now? $\mathcal{S}(k, l) \subset \mathcal{S}(k+1, l-1)$

**Aini:** $T_{a_1 \ldots a_k}^{b_1 \ldots b_l}$ demand implicitly that

$T$ transforms as a tensor of type $\mathcal{S}(k, l)$ under change of coordinate basis.

$T \in \mathcal{S}(k, l)$ is really in $\mathcal{A}_{k, l}$ an infinite list of tensors, one for each pair $p \in M$.

Outer products: $\forall T_{a_1 \ldots a_k}^{b_1 \ldots b_l} \in \mathcal{S}(k, l) \& S_{c_1 \ldots c_m}^{d_1 \ldots d_n} \in \mathcal{S}(m, n)$
Outer products: \( T^{a_1 \ldots a_n} b \in \mathcal{J}(n,1) \) & \( s^{i_1 \ldots i_k} \in \mathcal{J}(k,1) \)

Then \( A \in N \) for \( \text{trace outer product} \) defined to be:

\[
T^{a_1 \ldots a_n} b \cdot s^{i_1 \ldots i_k} \in \mathcal{J}(n+k,1)
\]

**Metric:** \( g \in \mathcal{J}(0,2) \) then \( A \in N \):

\[
g_{ab}
\]

The inverse of \( g \):

Form its metric inverse \( g^{-1} \in \mathcal{J}(2,0) \)

\[
(g^{-1})^{ab} = g_{ba} \in \mathcal{J}(2,0)
\]

**Raising/lowering:** Apply \( g \) to vector \( \mathbf{v} \):

\[
g \rightarrow g_{ab} \quad \mathbf{v} \rightarrow g \mathbf{v}
\]

(1) Outer product:

\[
g_{ab} v^c
\]

(2) Covariant:

\[
g_{ab} v^c \rightarrow \mathbf{e}_{a}, \quad g_{ab} v^c \in \mathcal{J}(0,1)
\]

\[
A \in N: \quad v^a \rightarrow g_{ab} v^c = \nabla_a \mathbf{v} \in \mathcal{J}(0,1)
\]

\[
\mathbf{v}^a \rightarrow \nabla_a \mathbf{v} \quad (\text{really stands for } g_{ab} \nabla_a v^b)
\]

**Lemma:**

\[
g^{ab} g_{bc} = \delta^a_c
\]

Proof: \( \Rightarrow e_{a} (g^{-1} \otimes g) \quad \Rightarrow g^{-1} g = I \)

\[
I = A \in N: \quad \delta^a_c
\]

**Generalized raising/lowering:** Let \( T^{abc} \in \mathcal{J}(3,3) \)

\[
A \in N: \quad g_{aa'} T^{abc} \Rightarrow T_{a'bc} \Rightarrow \mathcal{J}(2,4)
\]
\[ g_{ab} \cdot T^{bc} \text{ def } = T_{a}^{bc} \text{ def } = 1 \begin{pmatrix} 2 & 4 \end{pmatrix} \]

Notation is consistent with repeated applications of \( g \) & \( g^{-1} \).

\[ T^{abc} \text{ def } = g_{a}^{a'} g_{b}^{b'} T^{d'bc} \text{ def } = S_{a}^{a'} T^{d'bc} \text{ def } = T^{abc} \text{ def } \]

**Subspaces of symmetric / anti-symmetric tensors**

If \( T, T' \in \mathcal{S}(k) \),

\[ T + T' \in \mathcal{S}(4,k) \]

\[ T_{a}^{\cdots a} u_{b} \cdots b_{c} + T_{a}^{\cdots a} u_{b} \cdots b_{c} \]

**Definition**

\[ T_{(ab)} \equiv \frac{1}{2} \left( T_{ab} + T_{ba} \right) \]

\[ T_{(ab)} \equiv \frac{1}{2} \left( T_{ab} - T_{ba} \right) \]

\[ T_{a b c} \text{ def } = T^{a b c} \]

\[ T_{b a c} \text{ def } = T^{b a c} \]

**Definition**

**for** \( T_{a_{n-1} \cdots a_{2}} \in \mathcal{S}(0,n) \):

\[ T_{\{a_{1} \cdots a_{n}\}} \equiv \frac{1}{n !} \sum_{\pi \in S(n)} T_{a_{\pi 1} \cdots a_{\pi n}} \]

\[ T_{\{a_{1} \cdots a_{n}\}} \equiv \frac{1}{n !} \sum_{\pi \in S(n)} E(\pi) T_{a_{\pi 1} \cdots a_{\pi n}} \]

\[ E(\pi) = \begin{cases} 1 & \text{ if } \pi \text{ is a product of } \sigma \text{ and } \sigma^{-1} \text{ of transpositions} \\ 0 & \text{ otherwise} \end{cases} \]

**Commutator notation**:

\[ [a_{1} b_{1} \cdots a_{n} b_{n}] = \frac{1}{n!} \left( T_{a_{1} b_{1} \cdots a_{n} b_{n}} + T_{a_{1} b_{1} \cdots a_{n} b_{n}} - T_{a_{1} b_{1} \cdots a_{n} b_{n}} - T_{a_{1} b_{1} \cdots a_{n} b_{n}} \right) \]

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A totally antisymmetric tensor of type $(0,2)$ is called a differential 2-form.

Curvature: Spacetime is not embedded.

We want an intrinsic notion of curvature.

If $\mathcal{M}$ were embedded in $\mathbb{R}^k$, it is "easy" to see if $\mathcal{M}$ is curved.

We need a proxy for curvature: capture curvature by noticing that in flat manifolds we can move vectors around "in a parallel fashion" independent of path chosen.

On curved embedded manifold, eg surface of sphere.

Curvature = dependence on path chosen of parallel transport.

Parallel transport: Let $\mathcal{M}$ be a manifold with no additional structure. It turns out to be impossible.
to define a natural notion of parallel transport

Problem: we want to move a vector from \( V_p \) at \( p \) to \( V_q \) at \( q \) in a "parallel in way as possible".

\( \Rightarrow \) no natural way to compare elements of \( V_p \) and \( V_q \).

(Or even of \( V_p \) and \( V_{p+\delta p} \)).

In Euclidean space \( \mathbb{R}^n \) we take a vector at \( p \) and
swiff (using additive structure) to \( q = p + \delta p \) allows
us to define differences of vectors \( \nabla^p \frac{\partial}{\partial x^1} \equiv v_p \)

\[
\frac{\partial}{\partial x^1} \xrightarrow{\delta x^1} \nabla^p \left( \cdots , x^1 + \delta x^1, \cdots \right) - \nabla^p \left( \cdots , x^1 \right)
\]

defined at \( V_q = p + \delta p \)

We assume parallel transported vector of \( v \rightarrow q \)
has same components as \( v \) at \( p \).

Heuristic approach:

We need additional data, a parallel transporter \( U_q: V_p \rightarrow V_q \).

where \( U_q \) is a smooth path connecting \( p \) and \( q \).

If you had the data of a parallel transporter \( U_q \)
you can compare tangent vectors at \( p \) and \( q \).

Let \( v \in V_p \), \( w \in V_q \) then

define "\( v \) at \( q \)" to be

\[
U_q w \in V_q, \quad w \in V_q
\]
Demand $U_x$ is a linear transformation of vector spaces. What do we need to specify $U_x$?

\[ \Gamma U : (p, x - x') \times V \rightarrow V \]

Let's work infinitesimally: let $x \in V$.

Suppose $p$ has components $x^i$ in a chart $\psi$.

Consider $q$ near $p$ (infinitesimally close).

In chart $\psi$:

\[ \psi(q) = x + \Delta x \]

Let $Y$ be a smooth path connecting $p$ to $q$.

(\text{in chart } \psi: \psi \circ Y). Demand for

\[ \tilde{U} = U_x \circ Y \quad \text{that components satisfy} \]

(i) \[ \tilde{v}^m - v^m = \Delta x^m \]

(ii) \[ (\tilde{v}^m + v^m) = \tilde{u}^m + u^m \]

We can satisfy both (i) and (ii) if we take

(\text{w}):

\[ \tilde{v}^m = \hat{v}^m - \hat{v}^m \]

\[ \hat{v}^m = \Gamma_{x, x'} \]

\[ \Gamma_{x, x'} \quad \Delta x^m \]

\[ \text{original component} \]

\[ \text{connection coefficient} \]

\[ \hat{v}^m = v^m + \Delta x^m \]

\[ \text{we need these} \]

\[ \text{for each pair } x \text{ in chart} \]

\[ \text{chart} \]
Jet all this looks coordinate dependent; look for an \textit{intrinsic} way to define this.

\textbf{Note:} to every infinitesimal motion of parallel transport \( u^\alpha \Rightarrow \text{get derivative-type operator for vectors} \)

\[
\nabla_v \left( \frac{v^\mu}{\partial x^\nu} \right) = \lim_{\Delta x^\mu \to 0} \frac{v^\mu(x + \Delta x) - v^\mu(x)}{\Delta x^\nu} \frac{2}{\partial x^\nu} = \left( \frac{\partial v^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu \lambda} v^\lambda \right) \frac{2}{\partial x^\nu}
\]