Action of \((D_{a} D_{b} - D_{b} D_{a})\) on vectors & tensors may be obtained as follows:

Let \(t^{a}\) & \(\omega^{a}\) be an arbitrary vector & dual vector fields. Consider \(t^{a}\omega_{a}\):

\[
0 = (D_{a} D_{b} - D_{b} D_{a}) (t^{c} \omega_{c}) = D_{a} (\omega_{c} D_{b} t^{c} + t^{c} \omega_{b} t^{e} \partial_{e} \omega_{c}) - D_{b} (\omega_{c} D_{a} t^{c} + t^{c} \omega_{a} \partial_{e} \omega_{c})
\]

\[
= \omega_{c} (D_{a} D_{b} - D_{b} D_{a}) t^{c} + t^{c} (D_{b} D_{a} - D_{a} D_{b}) \omega_{c}
\]

\[
0 = \omega_{c} (\underbrace{- R_{abcd} t^{d}}_{\text{Riemann tensor}}) + t^{c} \underbrace{\omega_{d} R_{a d b}}_{\text{Ricci tensor}}
\]

\[
(D_{a} D_{b} - D_{b} D_{a}) t^{c} = - R_{abcd} t^{d}
\]

Similarly, ex1: we find \(T^{ab}_{\Lambda a} b \Lambda \):

\[
(D_{a} D_{b} - D_{b} D_{a}) T^{c_{1} \ldots c_{i}}_{d_{1} \ldots d_{j}} = - \sum_{j=1}^{i} R_{abcd} T^{c_{1} \ldots c_{i-1} d_{j}}_{d_{1} \ldots d_{j-1}} + \sum_{j=1}^{i} R_{abcd} T^{c_{1} \ldots c_{i} d_{j}}_{d_{1} \ldots d_{j-1}}
\]

The Riemann tensor has four important properties:

1. \(R_{abcd} = - R_{bacd}\)
2. \(R_{abcd} = 0\)
3. For \(\nabla_{a}\) defined by metric \(\nabla_{a} g_{bc} = 0\):
   \(R_{abcd} = - R_{bacd}\)
4. \(\nabla_{a} R_{bcd}^{e} = 0\) (Bianchi identity)

Proof: 1 follows by definition.

2. Note for arbitrary \(\omega_{a}\) & \(\nabla_{a}\) we have

\[
\nabla_{a} \nabla_{b} \omega_{c d} = 0
\]

Check this as follows:

\[
\text{LHS:}
\]

\[
= \nabla_{a} \nabla_{b} \omega_{c d} + \nabla_{a} C_{C \Lambda} \omega_{c d} + \nabla_{a} \omega_{c d}
\]

\[
= \underbrace{\nabla_{a} \nabla_{b} \omega_{c d}}_{0} + C_{C \Lambda} \nabla_{a} \omega_{c d} + C_{C \Lambda} \omega_{c d}
\]

\[
= - \underbrace{\nabla_{a} \nabla_{b} \omega_{c d}}_{0} + \underbrace{C_{C \Lambda} \nabla_{a} \omega_{c d}}_{0} + 
\]

\[
= 0
\]

Antisymmetric \(c\) over \(a b c\), leave \(d\) alone.
\[ + \partial_a \varepsilon^d \omega_a + C_{abc} \varepsilon^{[c} \omega_c \varepsilon^{d] \omega_a} + C_{[abc] \varepsilon^{d]} \omega_a \varepsilon^d \omega_b \varepsilon^c \omega_d = 0 \]

\[ 0 = 2 \nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = R_{abcd} \omega_d \]

3. Consider:

\[ 0 = (\nabla_a \nabla_b - \nabla_b \nabla_a) \omega_c \frac{1}{e^a} R_{abcd} \omega_d + R_{ab} \omega_c = R_{abcd} \omega_d + \nabla_a \omega_c \]

4. Consider:

\[ (i) = (\nabla_a \nabla_b - \nabla_b \nabla_a) \nabla_c \omega_d = R_{abcd} \omega_d + R_{ab} \nabla_c \omega_d \]

\[ (ii) = \nabla_a (\nabla_c \omega_d - \nabla_d \omega_c) = \nabla_a (R_{bcd} \omega_e) = \omega_c \nabla_a R_{bcd} \omega_e + \omega_d \nabla_a \omega_c \omega_e \]

Annihilate (i) & (ii) over all \( a \) \( \Rightarrow \) both expressions become equal:

\[ R_{abcd} \omega_d + R_{[ab]d} \nabla_c \omega_f = \omega_e \nabla[a R_{bce}] \omega_e + \omega_e \nabla[c R_{abe}] \omega_e \]

Thus, for all \( a \), \( \nabla_c R_{abcd} \omega_d = 0 \)

\[ \textbf{Riemann Curvature Tensor:} \]

This argument defines \( 2 \) 2nd derivative of \( 1 \) coordinate of \( 2 \) vector \( \omega \). This is the definition of 2nd derivative of \( 2 \) coordinate \( \partial_x \) of \( \omega \).

\[ R(\omega_a \omega_b \omega_c \omega_d) = M_{[a \beta \omega c \omega d]} \]

\[ \omega \times \omega \text{ has components} \]
\[ R(v, w, \ldots) = M_{a}^{b} \]

**Dehn's**

**Ricci tensor:**
\[ R_{ac} = R_{abc}^b \]

**Note:**
\[ R_{ab} = R_{ba} \] (by \( \mathcal{L} \))

**Scalar curvature:**
\[ R = R_{a}^{a} \]

**Contract Bianchi identity:**
\[ \nabla_{a} R_{bcd} + \nabla_{b} R_{cda} - \nabla_{c} R_{dab} = 0 \]

**Contract with \( g^{bd} \):**
\[ \nabla_{a} R_{c}^{c} + \nabla_{b} R_{c}^{b} - \nabla_{c} R = 0 \]

Written
\[ \nabla_{a} G_{ab} = 0 \] ("Divergence")

**Where:**
\[ G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \]

Is the **Einstein tensor**.

**Geodesic deviation equation:**

Tells us how nearby geodesics accelerate away or toward each other.

Suppose we have a smooth one-parameter family \( \gamma_{s}(t) \) of geodesics. That is, \( \gamma_{s} \) is a geodesic \( \forall s \in \mathbb{R} \), and \( (s, t) \longrightarrow \gamma_{s}(t) \) is smooth, one-to-one map with smooth inverse. Let \( \Sigma \) be a 2-dimensional submanifold described by \( \gamma_{s}(r) ; (s, r) \) are a good system of \( \Sigma \).
described by $\gamma_s(t)$; $(s,t)$ are a coord system of $\mathfrak{I}$

\[ T^a = (\partial^a s)^t \] is tangent to $\gamma_s(t)$, so we have

\[ T^a \partial_a T^b = 0 \]

The vector field $X^a = (\partial^a s)^t$ is displacement to infinitesimally close geodesic $\gamma_{st}(t)$

Recall + with a $s$-dependent factor so that

\[ T^a \partial_s T^b \]

does not vary with $s$. Since $X^a, T^a$ are coordinate fields so

\[ T^b \partial_b X^a = X^b \partial_b T^a \]

\Rightarrow \quad X^a T_a = \text{constant}. \quad \text{A reparametrisation by addition of (s-dependent) constant}

\[ X^a T_a = 0 \]

The quantity $v^a = T^a \partial_s X^a$ measures rate of change of displacement $X^a$ along geodesic; "relative velocity" relative acceleration

\[ A^a = T^c \partial_c v^a = T^c \partial_c \left( T^b \partial_b X^a \right) \]

\[ = T^c \partial_c \left( X^b \partial_b T^a \right) \hspace{1cm} (X^a \& T^a \text{ conn.)} \]

\[ = (T^c \partial_c X^a) (\partial_b T^a) + X^b T^c \partial_c \partial_b T^a \]

\[ = (X^c \partial_c T^b) (\partial_b T^a) + X^b T^c \partial_c T^a - R_{cba} T^c T^a \]

\[ = X^c \partial_c (T^b \partial_b T^a) - R_{cba} T^c T^a \]

\[ = -R_{a(cba)} T^c T^a \]

\Rightarrow \quad \text{Geodesic deviation equation}

\[ \Rightarrow \quad A^a = 0 \quad \text{for all geodesic families } \gamma_s(t) \implies \mathcal{R}_{abc} = 0 \]
If \( \exists \text{ relatively accelerating geodesics (familiar)} \)
\[
\Rightarrow R_{abcd} = 0
\]

Computing curvature

Given manifold \( \mathcal{M} \) with metric \( g_{ab} \) then \( R_{abc} \) is
determined uniquely by \( \nabla_a \)

In practice we choose coordinates \( (y_1, x^a) \)
determined by \( y \).

\[
\nabla_b \omega_c = \nabla_b \omega_c - \Gamma^d_{bc} \omega_d
\]

So

\[
\nabla_a \nabla_b \omega_c = \nabla_a (\nabla_b \omega_c - \Gamma^d_{bc} \omega_d) - \Gamma^d_{ab} (\omega_c - \Gamma^e_{ec} \omega_e) - \Gamma^d_{ac} (\omega_c - \Gamma^e_{ec} \omega_e)
\]

Hence,

\[
R_{abc} \omega_d = 2 \nabla_a \nabla_b \omega_c \quad \text{(ex.)}
\]

\[
R_{abc} \omega_d = \left( -2 \, \partial_a \Gamma^d_{bc} + 2 \, \Gamma^e_{ce} \Gamma^d_{ec} \right) \omega_d
\]

This holds for all \( \omega_d \), so in coordinates \( y \):

\[
R_{\mu \nu \rho \sigma} = -\frac{2}{\partial^x} \Gamma_{\rho \sigma}^{\nu} - \frac{2}{\partial^x} \Gamma_{\nu \mu}^{\rho} + \sum_{a} \left( \Gamma_{\rho \nu}^{a} \Gamma_{\sigma \mu}^{a} - \Gamma_{\rho \mu}^{a} \Gamma_{\sigma \nu}^{a} \right)
\]

Ricci tensor is simply \( R_{\mu \rho} = \sum_{\nu} R_{\mu \nu \rho} \)

Some useful formulas

Define \( g = \det (g_{\mu \nu}) \)

Use formula for inverse of matrix: \( (ex) \)

\[
\sum_{\nu, \mu} g^{\nu \mu} \frac{\partial g_{\mu \nu}}{\partial x^\alpha} = -\frac{1}{2} \frac{\partial g}{\partial x^\alpha}
\]

and

\[
\partial \log |g| = \log |g|\]
This equation appears via formula for Christoffel symbols

\[ \Gamma^\alpha_{\beta\gamma} = \sum_v \Gamma^v_{\beta\gamma} = \frac{1}{2} \frac{\partial \Gamma^v}{\partial x^\gamma} + \frac{1}{2} \frac{\partial \Gamma^\gamma}{\partial x^\beta} - \frac{1}{2} g^\gamma_r \frac{\partial g_{v\alpha}}{\partial x^\beta} \]

\[ = \frac{1}{2} \frac{1}{g^{\alpha\beta}} \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \]

Appears as well in divergence \( \nabla_a T^a \) of vector field \( T^a \).

\[ \nabla_a T^a = \partial_a T^a + \Gamma^a_{\beta\gamma} T^\beta \]

\[ = \frac{1}{2} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^a} \left( \sqrt{|g|} T^a \right) \]

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