Length of $c$ (w.r.t. $g_{ab}$):

$$L = \int (g_{ab} T^a T^b)^{1/2} \, dt$$

where $T^a$ is the tangent to $c$, $t$ is curve parameter.

I assume that our metric $g$ has signature $(+,+,\ldots,+)$.

For metric with signature $(-,+,\ldots,+)$, a curve $c$ is said to be **timelike** if

$$g_{ab} T^a T^b < 0$$

everywhere along curve; **null** if

$$g_{ab} T^a T^b = 0$$

and **spacelike** if

$$g_{ab} T^a T^b > 0$$

Everywhere along $c$.

For timelike curve $c$ define the proper time $\tau$ via

$$\tau = \int \sqrt{-g_{ab} T^a T^b} \, dt$$

If a curve $c$ changes from eg, timelike $\Rightarrow$ spacelike,

⇒ length is not defined.

Since, for a geodesic the tangent vector $T^a$ is parallel transported along curve itself, its norm

$$(T^a, T^b) = g_{ab} T^a T^b$$

cannot change ⇒ a geodesic cannot change from null to timelike etc.
\( \text{Change in norm:} \quad (T^a \nabla_a) \text{ norm} = (T^a \nabla_a)(g_{ab} T^b T^a) \)
\[ = \left( T^a \nabla_a g_{ab} \right) (T^b T^a) + g_{ab} \left( T^a \nabla_a T^b \right) + g_{ab} T^a \nabla_a T^b \]
\[ = 0 \]

\( \text{L (k=1) is reparameterization invariant:} \)

**Suppose we chose \( \text{SLT} \), instead of \( t \), to parameterize**

C. New tangent vector (ex.)

\[ s^a = \frac{dt}{ds} T^a \]

The length \( L \) defined w.r.t. \( s \) is

\[ L = \int \sqrt{g_{ab} s^a s^b} ds = \int \sqrt{g_{tt} T^t T^t} \cdot \frac{dt}{ds} ds = L \]

Claim: geodesics extremize length of curve joining two points \( p \) and \( q \).

Suppose \( p, q \) are in a common chart \( \psi: \mathbb{M} \to \mathbb{R}^n \).

Assume \( M \) is spacelike.

\[ (\ell) = \int_a^b \sqrt{\sum_{\mu \nu} g_{\mu \nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} \, dt \]

where \( C(a) = p \), \( C(b) = q \) are end points.

Green curve is an infinitesimal variation of \( \psi(x) \)

\[ x^\mu(t) \quad \longrightarrow \quad x^\mu(t) + 8x^\mu(t) \]

where \( 8x^\mu(t) = 8x^\mu(t) - 0 \)

\[ \ell(x + 8x) = \ell(x) + 8\ell(t) \]

How does the length of \( t \) \( C = x + 8x \) change?

\[ (\delta\ell) = \int_a^b \left( \sum_{\mu \nu} g_{\mu \nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right)^{1/2} \left\{ \dot{\mu} \delta x^\mu + \frac{1}{2} \sum_{\mu \nu} \frac{\partial g_{\mu \nu}}{\partial x^\rho} \dot{\rho} \delta x^\mu \dot{\nu} \right\} \, dt \]

Assume we have chosen a parametrization which so that

\[ \frac{ds}{dt} \quad \text{is scalar} \]

\( \text{Curve} \quad \underbrace{\in}_{\psi} \quad \text{number} \)
Assume we have chosen a parametrization which so that
\[ g_{ab} T^a T^b = 1 \]

**Extremality:**
\[ S_t = 0 \]

\[ A \frac{\partial}{\partial x^a} \]

\[ 0 = - \sum_a g_{\mu \nu} \frac{d^2 x^\mu}{dt^2} - \sum_a \frac{\partial g_{\mu \nu}}{\partial x^a} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + \int \cdots \frac{\partial g_{\mu \nu}}{\partial x^a} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \]

\[ \Rightarrow \text{This is the geodesic equation.} \]

A curve extremizes the length \( \Rightarrow \) it is a geodesic.

A similar derivation shows that geodesic equation can be obtained by varying extremizing Lagrangian

\[ L = \sum_{\mu \nu} g_{\mu \nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \]

\[ \Rightarrow \text{This can give a very efficient way to calculate} \]

\[ \Gamma \left( \frac{d^2 x^\mu}{dt^2} \right) = 0 \]

---

**Extremum vs. maximum.**

For Lorentz signature manifolds \( M \) a curve (timelike) joining two points may have arbitrarily small proper time

\[ \text{has smaller proper time} \]

If a curve (timelike) of greatest proper time exists it
must be a timelike geodesic. Note a given geodesic for \( p \to q \) need not maximize proper time.

**Curvature**

Path-dependence of parallel transport of a vector from \( V_0 \to V_1 \) gives an intrinsic proxy for curvature of \( M \).

**Riemann curvature tensor** $\equiv$ measure of failure of successive parallel transport operations to commute.

Start by studying action $\nabla_a V_0$ on an arbitrary and vector field $\omega_c$. Suppose $f \in \mathcal{F}(\mathbb{R})$ & consider

$$
\nabla_a V_0 (f \omega_c) = \nabla_a (\omega_c \nabla_0 f + f \nabla_0 \omega_c) = (\nabla_a \nabla_0 f) \omega_c + (\nabla_0 f)(\nabla_a \omega_c) + (\nabla_0 (\nabla_a f)) \omega_c + f \nabla_a \nabla_0 \omega_c
$$

Consider

$$(\nabla_a \nabla_0 - \nabla_0 \nabla_a) (f \omega_c) = f (\nabla_a \nabla_0 - \nabla_0 \nabla_a) \omega_c \quad \text{(true for } f \in \mathcal{F}(\mathbb{R}))$$

Thus $(\nabla_a \nabla_0 - \nabla_0 \nabla_a) \omega_c$ only depends on value of $\omega_c$ at $p$,

So

$$(\nabla_a \nabla_0 - \nabla_0 \nabla_a) : (\omega_c) \to \mathcal{F}(\mathbb{R}),$$

We thus obtain $R_{abc}^d \in \mathcal{F}(\mathbb{R})$ such that $\forall \omega_c \in \mathcal{F}(\mathbb{R})$

$$R_{abc}^d \omega_c = (\nabla_a \nabla_0 - \nabla_0 \nabla_a) \omega_c$$

$R_{abc}^d \equiv$ Riemann curvature tensor.

**Relate** $R_{abc}^d$ to failure of successive parallel transport.

Consider a loop starting at $p \in M$ defined by a 2D surface $S$ through $p$. Let coordinates of $S$ be $(s^1, s^2)$ with $(s^1, s^2) \equiv (0, 0)$ at $p$. The loop is then defined by

$$(0, 0) \to (0, s^2) \to (0, s^1, s^2) \to (s^1, 0) \to (s^1, s^1, 0) \to (0, s^2).$$

Let $V^a$ be an arbitrary vector at $p$, and parallel transport it around loop. Suppose $\omega_c \in \mathcal{F}(0, 1)$ is arbitrary.
Let \( v^a \) be an arbitrary vector at \( p \) and parallel transport it around loop. Suppose \( \omega_a \in \mathcal{F}(0,1) \) is arbitrary.

Consider \( v^a \omega_a \in \mathcal{F}(0,1) \).

The change \( \delta_1 \) in \( v^a \omega_a \) along \( i_1 \) is

\[
\delta_1 = \Delta t \left( T^b \nabla_b (v^a \omega_a) \right)_{(\Delta t, 0)} \left( \Delta t, 0 \right)
\]

\[
= \Delta t \left( T^a v^a \nabla_a \omega_a \right)_{(\Delta t, 0)} \left( \Delta t, 0 \right) \quad (v^a \text{ is parallel})
\]

Here \( T^a \) is the tangent vector to curve with constant \( s \).

Similarly, we obtain

\[
\delta_3 = -\Delta t \left( T^a v^a \nabla_a \omega_a \right)_{(\Delta t, 0)} \left( \Delta t, 0 \right)
\]

Combine \( \delta_1 + \delta_3 \):

\[
\delta_1 + \delta_3 = \Delta t \left( T^a v^a \nabla_a \omega_a \right)_{(\Delta t, 0)} \left( (\Delta t, 0) \right)
\]

Note: \( \delta_1 + \delta_3 \to 0 \) as \( \Delta s \to 0 \). Also \( \delta_2 + \delta_4 \to 0 \) as \( \Delta s \to 0 \).

So \( \delta_1 + \delta_2 + \delta_3 + \delta_4 = 0 \) to first order in \( (\Delta s, \Delta t) \).

Parallel transport around loop is path-independent to first order.

To get 2nd order dependence consider parallel transport of \( v^a \) and \( T^b \nabla_b \omega_a \) along \( t = \Delta t \).

To first order \( v^a \) at \((\Delta s, \Delta t/2)\) is equal to \( v^a \) at \((0, \Delta t)\).

The quantity \( T^b \nabla_b \omega_a \) at \((\Delta s, \Delta t/2)\) differs from \( T^b \nabla_b \omega_a \) at \((0, \Delta t/2)\) parallel transported to \((\Delta s, \Delta t/2)\).
\[ \delta \frac{\partial}{\partial x^c} \]

where \( \delta^e \) is tangent to curve of constant \( t \). Substituting (\( \star \)) into \( \delta_1 + \delta_2 ):

\[ \delta_1 + \delta_2 = -\Delta t \delta v^a S^c \nabla_c (T^b \nabla_b \omega_a) \quad \text{(*)} \]

(\( \text{similar for } \delta_3 + \delta_4 \))

Add together (\( \star \))

\[ \delta (\omega_a) = \delta s \Delta t \quad 1 \quad \Delta t \quad \nabla^a (T^c \nabla_c (S^b \nabla_b \omega_a - \frac{S^b}{\nabla_b (T^c \nabla_c \omega_a - T^b \nabla_b \omega_c) \nabla_c \omega_a}) \]

\[ = \Delta t \quad \nabla^a \quad T^c \nabla_c (\nabla_c \omega_a - \nabla_b \omega_c) \nabla_c \omega_a \]

\[ = \Delta t \quad \nabla^a \quad 1 \quad \Delta t \quad \nabla^a \quad R^{a c b d} \quad \omega_d \]

Here we used the fact that

\[ \nabla^a \omega_a = \omega^a \nabla_a \omega^b - \omega^b \nabla_a \omega^b = 0 \]

for coordinate vector fields \( \omega^a = \frac{2}{\partial x^a}, \quad \omega^b = \frac{2}{\partial t} \)

This variation \( \delta (\omega_a) \) can hold for all \( \omega_a \) only if (to 2nd order in \( \Delta t \))

\[ \delta \omega^a = \Delta t \quad \nabla^a \quad T^c \nabla_c \quad R^{a c b d} \quad \omega_d \]