Introduction to general relativity

Equivalence principle implies it is impossible, in general, to set up a family of mutual observers to measure "gravitational force": We cannot "insulate" an observer from gravity!

**Fundamental hypothesis of general relativity:**

Spacetime is a manifold \( M \) on which is defined a Lorentzian metric \( g_{ab} \). (\( M \) may be much more general than \( R^{1,3} \))

Further, spacetime is not necessarily flat. The world lines of freely falling bodies are geodesics of (curved) metric \( g_{ab} \).

**Note:** In case of time-translation symmetry, one can set up a preferred background family of observers and "measure" a "gravitational force" w.r.t. family.

While we cannot measure gravitational force in general situations, we can measure relative acceleration of nearby geodesics \( \Rightarrow \) we speak of tidal forces.

Two key principles to determine laws of physics \( \Rightarrow \)

1. **General covariance:** The metric \( g_{ab} \) (and quantities derivable from \( g_{ab} \)) are the only spacetime quantities that can appear in laws of physics (equations).

2. **Equations of physics must reduce to their SR versions for \( g_{ab} \equiv \text{flat} \)

According to (1) and (2) we continue to represent physical quantities via some tensorial quantities as in SR. Thus:

*1 particle motion is represented by timelike curve \( C \)
(k) The 4-velocity of a particle is the unit tangent to its worldline: \( u^a \) (measured w.r.t. \( g_{ab} \)).

We need to amend equations of motion:

Free-particle equation of motion is its geodesic equation:

\[
\n^a \nabla_a u^b = 0
\]

where \( \nabla_a \) is determined by \( g_{ab} \). Acceleration is defined analogously:

\[
a^b = u^a \nabla_a u^b
\]

when \( a \neq 0 \) we say a force

\[
f^b = ma^b
\]

where \( m \) is rest mass of particle.

The 4-momentum of the particle is defined as

\[
p^a = m u^a
\]

The energy, as determined by observers present at the particle's worldline at which the energy measured is

\[
E = -p_a u^a
\]

where \( u^a \) is velocity of observer.

A given observer cannot define energy of distant particle because parallel transport is path-dependent.

“GR-friendly” equations of motion can be found (usually) by applying “minimal substitution” rules

\[
SR: \quad \eta_{ab} \quad \rightarrow \quad GE: \quad g_{ab}
\]
SR: $\eta_{ab}$ $\longrightarrow$ GR: $g_{ab}$

SR: $\delta_a$ $\longrightarrow$ GR: $\nabla_a$

Examples: the Klein–Gordon field (in SR)

$\mathcal{L} = \frac{1}{2} (\partial_a \phi)(\partial^a \phi) - \frac{1}{2} m^2 \phi^2$

$\Rightarrow$ equation of motion

$\partial_a \partial^a \phi - m^2 \phi = 0$

In GR: $\partial_a \longrightarrow \nabla_a$

Not only generalization possible!

$\nabla_a \nabla_a \phi - m^2 \phi = 0$

Stress-energy tensor in GR

$T_{ab} = \partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab} (\partial_c \phi \partial^c \phi + m^2 \phi)$

$\nabla^a T_{ab} = 0$

Example (perfect fluid in SR)

pertains to a continuous distribution of matter with Stress-Energy tensor $T_{ab}$ of form

$T_{ab} = \rho u_a u_b + P (g_{ab} + u_a u_b)$

where $\rho$ is density, $P$ is pressure field, $u^a$ is a unit 4-velocity vector field representing 4-velocity of fluid. Equation of motion of ideal perfect fluid:

$\nabla^a T_{ab} = 0$

In GR:

$T_{ab} = \rho u_a u_b + P (g_{ab} + u_a u_b)$

Impose (guess) equation of motion

$\nabla^a T_{ab} = 0$
By projecting onto components parallel & perpendicular to $u^\alpha$:

$$ u^\alpha \nabla_\alpha p + (\gamma + \rho) \nabla^\alpha u_\alpha = 0 $$

$$(\rho + p) u^\alpha \nabla_\alpha u_\alpha + (\gamma \rho + \rho \gamma) \nabla^\alpha p = 0$$

Comment on interpretation of Tab $v^\alpha$ in GR.

In GR: An observer with $+\text{-velocity } v^\alpha$ interacts:

$$ \text{Tab } v^\alpha v^\beta $$

as energy density, i.e. mass-energy density/unit volume for the observer. (Ex.) Further, if $x^\alpha$ is orthogonal to $v^\alpha$ then

$$ -\text{Tab } v^\alpha x^\alpha $$

is interpreted as momentum density of matter in $x^\alpha$ direction.

So, in GR, $\Theta^\alpha \text{Tab } = 0$ may mean be interpreted as conservation law, e.g. apply Gauss law in following situation. We assume we can set up a family of inertial observers with parallel $4$-velocities $u^\alpha$, so $\Theta^\alpha \varepsilon_{\alpha} = 0$

Define

$$ J_\alpha = -\text{Tab } v^\beta $$

So

$$ \Theta^\alpha \text{Tab } = 0 \Rightarrow \Theta^\alpha J_\alpha = 0 $$
In GR: A family of observers is represented by \( v^a \) (unit 4-velocity). The condition that 4-velocities are parallel:

\[ \nabla^a v_a = 0 \]

equivalently,

\[ \nabla_a (v_b) = 0 \] (Killing's equation)

However, in curved spacetime \( v^a \) is generally impossible to find \( v^a \) such that

\[ v^a v_a = -1 \]

\[ \nabla_a (v_b) = 0 \]

Counterexample: de Sitter spacetime.

Therefore \( \nabla^a T_{ab} = 0 \) does not imply strict global energy conservation.

Physically, the gravitational field can do work on fluid via tidal forces.

Can only regard \( T_{ab} \) as a local conservation of material energy, valid in small regions of spacetime.

Einstein's equation: spacetime is dynamical in GR.

We need equation of motion for matter or gas.

Mach's principle: spacetime geometry is influenced by matter distribution.

To make this quantitative we look at how tidal forces are calculated in Newtonian physics and GR.
Newton: gravitation field described by potential $\phi$, tidal acceleration vector $a$

$$a = - (x \cdot \nabla) \nabla \phi$$

where $x$ is relative separation vector.

In GR: tidal acceleration described by geodesic deviations

$$a^a = -R^a_{\phantom{a}cd} v^c x^d v^d$$

where $v^a = t$-velocity of particles and $x^i$ the deviation.

Strongly suggests correspondence:

$$R^a_{\phantom{a}cd} v^c v^d \leftrightarrow \partial_0 \partial^a \phi$$

However $\phi$ is determined by $\rho$ according to Poisson's equation

$$\nabla^2 \phi = 4\pi \rho$$

Energy-density of matter is described by stress-energy tensor

$$T^a_{\phantom{a}b} v^a v^b \leftrightarrow \rho$$

Thus: this suggests

$$R^a_{\phantom{a}cd} v^c v^d = 4\pi T_{cd} v^c v^d$$

$$\Rightarrow$$

$$R_{cd} = 4\pi T_{cd}$$

This was Einstein's original equation. However it has a flaw: namely

$$\nabla^a T_{ab} \neq 0$$

because of Bianchi identity:

$$\nabla^c (R_{cd} - k e_d e^c) = 0$$

We would need
Resolution: consider instead, the equation

\[ g_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab} \]

This is Einstein's equation.