Einstein's field equations, linearised solutions

\[(*) \quad R_{ab} - \frac{1}{2} R g_{ab} = 8 \pi T_{ab} \]

Basic properties:
- Take trace of (*)
  
  \[ R = -8 \pi T \]

Here \( T \) is trace of \( T_{ab} \). Substitute back into (*) so that

\[ R_{ab} = 8 \pi \left( T_{ab} - \frac{1}{2} g_{ab} T \right) \]

Geometry \( \rightarrow \) Energy-Matter-Stress etc.

Sometimes this equation can yield solutions to (*)

Do we actually recover, e.g., SR, and Newtonian physics as solutions to (*) or some limits?

Goal for next couple of lectures: Argue that GR reduces to SR/Newtonian in correct limits.

In case where energy of matter as measured by an observer "at rest" with respect to masses is greater than stresses (\( \mu \) units, where \( c=1 \)) then

\[ T^{\alpha \beta} - g^{\alpha \beta} T = -T_{ab} u^a u^b \]

So that (*)

\[ R_{ab} u^a u^b = 4 \pi T_{ab} u^a u^b \]
The physical content of GR is thus

Spacetime $\mathcal{M}$ is a manifold with Lorentzian metric $g_{ab}$. The curvature of $\mathcal{M}$ is related to its matter distribution via $\mathcal{R}_{\alpha\beta\gamma\delta}$.

1. Equation (1) is a highly nonlinear system of coupled PDEs involving $g_{uv}$ (in some coordinate system). It is 2nd order in derivatives.

2. Unlike, eg, Maxwell's equations, we cannot regard $T_{\alpha\beta}$ as a "source" and solve for $g_{\alpha\beta}$! This is because $T_{\alpha\beta}$ is defined in terms of $g_{\alpha\beta}$ for eg. Perfect fluids. We can't even interpret $T_{\alpha\beta}$ before solving (1).

3. Einstein's field equations imply $\nabla^{\alpha} T_{\alpha\beta} = 0$. It may be argued that $\nabla_{\alpha} T^{\alpha\beta} = 0$, for a "dust" of gravis exerting no force on each other, implies that gravis move on geodesics so, effectively (ii) contains geodesic hypersurfaces.

Linearised gravity

We now attempt solve (1) in case where gravity is "weak": ie, spacetime metric $g_{\alpha\beta}$ is nearly flat, which we assume means

$$g_{\alpha\beta} = \mathcal{g}_{\alpha\beta} + \xi_{\alpha\beta}$$

with $\xi_{\alpha\beta}$ "small".

In case of linearised gravity we say $\xi_{\alpha\beta}$ is small if there exists a global minimal coordinate system.
in case of nonvanishing gravity we say \( \eta \) is small if there exists a global minimal coordinate system such that \( \gamma_{\mu\nu} \) are small w.r.t. \( \eta \).

Linearized gravity is what results from substituting

\[ g^{ab} = \eta^{ab} + \delta g^{ab} \]

into (1) (and collecting terms to first order in \( \gamma_{\mu\nu} \)).

We raise & lower indices with \( \eta^{ab} \); except for \( g^{ab} \) denotes the inverse to \( g_{\mu\nu} \) (not \( \eta^{ab} \eta_{bc} g_{cd} \)).

Calculate \( g^{\alpha\beta} = \eta^{\alpha\beta} - \gamma^{\alpha\beta} \).

\[
\Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2}(\eta^{\alpha\gamma} - \gamma^{\alpha\gamma})(\partial_\alpha \gamma_{\beta\gamma} + \partial_\beta \gamma_{\alpha\gamma} - \partial_\gamma \gamma_{\alpha\beta})
\]

In global coordinate system, to linear in \( \gamma_{\mu\nu} \), we find

\[
\Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2}(\eta^{\alpha\gamma} - \gamma^{\alpha\gamma})(\partial_\alpha \gamma_{\beta\gamma} + \partial_\beta \gamma_{\alpha\gamma} - \partial_\gamma \gamma_{\alpha\beta})
\]

Thus to linear order in \( \gamma_{\mu\nu} \), Ricci tensor is (ex.)

\[
R_{ab} = \partial_c \Gamma_{ab} - \partial_a \Gamma_{cb}
\]

\[
= \partial^c \partial_c (\gamma_{ab}) - \frac{1}{2} \partial_c \partial_c \gamma_{ab} - \frac{1}{2} \partial_a \partial_b \gamma_{cc}
\]

where \( \gamma = \gamma_{cc} \).

\[
^{(1)} g^{ab} = R^{(1)}_{ab} - \frac{1}{2} \eta^{ab} R^{(1)}
\]

\[
^{(2)} g^{ab} = \partial^c \partial_c (\gamma_{ab}) - \frac{1}{2} \partial_c \partial_c \gamma_{ab} - \frac{1}{2} \partial_a \partial_b \gamma_{cc} - \frac{1}{2} \eta_{ab} (\partial^c \partial_c \gamma_{cc} - \frac{1}{2} \partial^c \partial_c \gamma_{cc})
\]

If we define \( \gamma^{ab} = \gamma_{ab} - \frac{1}{2} \eta_{ab} \gamma \), then (2) to linear order in \( \gamma \) become (1).
\[
\frac{1}{2} \varepsilon_{de} \varepsilon_{fabc} + \varepsilon_{de} \varepsilon_{(fabc)} - \frac{1}{2} N_{ab} \varepsilon_{efd} F_{cd} = 8 \pi T_{ab}
\]

This equation may be simplified using the gauge freedom of general relativity: if \( \phi : M \to M \) is a diffeomorphism between two manifolds and \( g_{ab} \) represent the same spacetime geometry, this will be explained below.

**Maps of manifolds**

Let \( M \) and \( N \) be manifolds. Let \( \phi : M \to N \) be a \( C^\infty \) map. Associated to any function \( f : N \to \mathbb{R} \), there is a "pullback" function:

\[
\phi^* f : M \to \mathbb{R}
\]

To any tangent vector at \( p \in M \), there is a "pushforward" vector defined via

\[
\phi_* V_p : = V_{\phi(p)}
\]

where \( \phi_* \) is obtained as follows. For \( v \in V_p \), define \( \phi_* v \in V_{\phi(p)} \) by (e.g., check it is a tangent vector)

\[
(\phi_* v)(f) = v(\phi \circ f)
\]

for all \( f \in \mathfrak{X}(N) \).

The map \( \phi_* \) is given by Jacobian of \( \phi \). Let \( x^i \) denote coordinates of \( M \) (resp. \( y^i \) coordinates of \( N \)).
Then matrix of \( \phi^* \) is\( (\mathbb{C}, \mathbb{R}) \):

\[
(\phi^*)^\mu \nu = \frac{\partial y^\mu}{\partial x^\nu}.
\]

Similarly, we can “pullback” dual vectors at \( \phi(p) \) by defining \( \phi^* : V^*_p \rightarrow V^*_0 \) so that for all \( \nu \in V^*_0 \)

\[
(\phi^* \nu)^a = \nu^a (\phi + \nu)^a.
\]

Extend to tensors via \( J(0, 1) \) via:

\[
(\phi^* T)_{i_1 \cdots i_k}^{a_1 \cdots a_l} \nu^{a_1} \cdots \nu^{a_l} = T_{i_1 \cdots i_k}^{a_1 \cdots a_l} (\phi^* \nu)^{a_1} \cdots (\phi^* \nu)^{a_l}
\]

and to \( J(1, 0) \) via:

\[
(\phi^* + \nu)_{i_1 \cdots i_k}^{a_1 \cdots a_l} \nu^{a_1} \cdots \nu^{a_l} = T_{i_1 \cdots i_k}^{a_1 \cdots a_l} (\phi^* \nu)^{a_1} \cdots (\phi^* \nu)^{a_l}.
\]

We do not get an action on mixed tensors \( T \) because arrows go in wrong way. However, if \( \phi \) is diffeomorphism (true \( m=n \)) we can exploit \( \phi^{-1} \) to build/extend \( \phi^* \) to mixed tensors (because \( (\phi^{-1})^* : V^*_0 \rightarrow V^*_p \)).

We define:

\[
(\phi^* T)_{a_1 \cdots a_l} \nu^{a_1} \cdots \nu^{a_l} (\phi^* \nu)^{a_1} \cdots (\phi^* \nu)^{a_l} = T_{a_1 \cdots a_l} (\phi^* \nu)^{a_1} \cdots (\phi^* \nu)^{a_l}.
\]

Note \( (\mathbb{C}, \mathbb{R}) \): \( \phi^* = (\phi^{-1})^* \).
Definition: a diffeomorphism $\phi: M \rightarrow N$ such that $(\phi^*g_{ab}) = g_{ab}$ is called an isometry.